Y12 to Y13 Summer Workbook
Jack Hunt School
2024

Hi soon to be Y13s!

This booklet is to set out our expectations of you during the summer break. It is really important you are as prepared as possible for Y13 and so we have organised a pack of Summer Work to be completed.

You can access all the summer work from the Y12 Summer Work folder in OneNote. All students need to complete the Booster packs on Differentiation, Integration and Exponentials/Logs. Additionally, as we continue to develop independent learners, we expect you to use your QLAs from the Mock exams to select and complete any other Booster Packs for topics you underperformed in.

All online section tests for year 12 and the year 13 topics already completed this term should have at least one attempt completed. You will also need to complete or improve the relevant online section tests for each additional booster pack on Integral, to support you in this you may wish to try the Integral Assessment for each booster pack, the compulsory topics above already have a page with the assessment added. If you prefer the Interactive books on Integral to reading through the notes and examples, you can do these instead.

These will all be reviewed in September to ensure you have done sufficient work to place you in the best position prior to starting Year 13.

We will also review your progress with year 13 **baseline assessments** in September, so please prepare to the best of your ability!

Any questions, please email Mrs Webb or Ms van der Ark.

Other materials

Summer is long, and you may find yourself wondering about the future and how different Maths is beyond A level. The following resources may provide some direction:

YouTube Channels:

- Numberphile
- 3blue1brown
- Stand Up Maths
- Veritasium
- TL Maths

Online Resources:

- Intermediate Mathematical Challenge Questions and Past papers on UKMT website
- Dot and boxes is a classic mathematical game that can be played online (or on paper!)
- Towers of Hanoi is a classic mathematical puzzle you can play online.
- The following mini game: https://ncase.me/trust/

Books:

- *Professor Stewart's Cabinet of Mathematical Curiosities* by Ian Stewart
  Alex's Adventures in Numberland: Dispatches from the Wonderful World of Mathematics by Alex Bellos
- *Things to Make and Do in the Fourth Dimension* by Matt Parker
- *Hello World: How to be Human in the Age of the Machine* by Hannah Fry

Section 1: Introduction to differentiation

Notes and Examples

These notes include sub-sections on:

What is differentiation?

In this section, you will be studying the relationship between the position of a point on a curve and the gradient of the curve.

Straight lines are, by definition, lines of constant gradient. Curves, on the other hand, have varying gradient – the gradient depends on whereabouts you are on the curve. Differentiation is the process of finding the gradient at any point on a curve from the equation of the curve.

Differentiation, together with its reverse process, called integration, form the branch of mathematics called calculus. The discovery of calculus (Made in the 17th century by Isaac Newton in England and, independently, by Gottfried von Liebnitz in Germany) was one of the most significant advances in the history of mathematics and science, and was crucial to unlocking the mathematical basis of our planetary system.

Differentiation is the process of finding the *gradient function*, or *derivative*, or *derived function*. Given an equation for y in terms of x, the gradient function or derivative is written $\frac{\mathrm{d}y}{\mathrm{d}x}$, and gives the gradient of the curve in terms of x.

Investigating gradients

You can investigate how the gradients of chords approach the gradient of a tangent using the Explore resource ***The gradient of a curve***. You can then go on to investigate the pattern in the value of the gradient at different points on a curve.

You can also try the ***Differentiation walkthrough***.

AQA AS Mathematics Differentiation 1 Notes and Examples

Rules for finding derivatives.

If y is a polynomial function (made up of powers of x), the following rules will enable you to find the derivative $\frac{dy}{dx}$:

- The derivative of x^n is nx^{n-1},
- The derivative of kx^n is knx^{n-1},
- The derivative of a constant is zero.
- The derivative of a sum (or difference) is the sum (or difference) of the derivatives

Example 1
Differentiate $y = 2x^3 - 5x^2 + 4$

Solution
The derivative of $2x^3$ is $2 \times 3x^2 = 6x^2$
The derivative of $5x^2$ is $5 \times 2x = 10x$
The derivative of 4 is 0.

So $\frac{dy}{dx} = 6x^2 - 10x$.

For further examples and practice, try the ***Differentiation skill pack***.

The next example involves an expression which is the product of two functions. You **cannot** differentiate this by differentiating each function separately and then multiplying the results, i.e. the derivative of a product of two functions is **not** the product of the derivatives! So with examples involving brackets, you will need to multiply out the brackets first. (There is a rule for differentiating the product of two functions, but you do not need to know this yet.)

Example 2
(i) Find the derivative of $y = (x-2)(x^2+1)$.
(ii) Hence find the gradient of the curve at the point (2, 0)
(iii) Find the coordinates of the points where the gradient is zero.

Solution
(i) $y = (x-2)(x^2+1) = x^3 - 2x^2 + x - 2$

$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 4x + 1$

(ii) Substituting $x = 2$ into the gradient function,

$$\frac{dy}{dx} = 3\times 2^2 - 4\times 2 + 1$$
$$= 5$$

so the gradient of the curve at (2, 0) is 5.

(iii) The gradient of the curve is zero when $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 3x^2 - 4x + 1 = 0$

$\Rightarrow \quad (3x-1)(x-1) = 0$

$\Rightarrow x = \frac{1}{3}$ and $x = 1$.

Now calculate the y coordinates for these values of x.

When $x = \frac{1}{3}$, $y = \left(\frac{1}{3} - 2\right)\left(\frac{1}{9} + 1\right) = -\frac{5}{3}\times\frac{10}{9} = -\frac{50}{27}$

When $x = 1$, $y = (1-2)(1^2+1) = -2$.

So the points on the curve with gradient zero are $(1, -2)$ and $\left(\frac{1}{3}, -\frac{50}{27}\right)$

The points where the gradient is zero are called the *turning points* or *stationary points* of the curve. You will look at such points in more detail in Section 2.

The next example involves the quotient of two functions (i.e. one function divided by another). As with products the derivative of a quotient is not the quotient of the derivatives. You need to divide the fraction first.

Example 3

Differentiate $\dfrac{3x^2 - 4x^4}{2x}$.

Solution

$$y = \frac{3x^2 - 4x^4}{2x} = \frac{3x^2}{2x} - \frac{4x^4}{2x} = \tfrac{3}{2}x - 2x^3 \quad \Rightarrow \quad \frac{dy}{dx} = \tfrac{3}{2} - 6x^2$$

Finding tangents and normals to curves

The gradient of a tangent to a curve at a particular point is the same as the gradient of the curve at that point. So to find the equation of a tangent to a curve, you first need to find the gradient m of the curve via differentiation. You can then substitute m and the coordinates (x_1, y_1) of the point on the curve into the formula:

$$y - y_1 = m(x - x_1)$$

AQA AS Mathematics Differentiation 1 Notes and Examples

Example 4

Find the equation of the tangent to the curve $y = 2x^3 - 3x$ at the point with x-coordinate 1.

Solution

$y = 2x^3 - 3x \Rightarrow \frac{dy}{dx} = 6x^2 - 3.$

To find the gradient of the tangent, first differentiate the equation of the curve.

At the point with x-coordinate 1

$\frac{dy}{dx} = (6 \times 1^2) - 3 = 3$

The gradient of the tangent is therefore 3.

Find the y-coordinate of the point where $x = 1$ by substituting into the equation of the curve.

$y = 2x^3 - 3x$

$x = 1 \Rightarrow y = (2 \times 1^3) - (3 \times 1) = 2 - 3 = -1$

Use the formula for the equation of a line with $m = 3$, $x_1 = 1$ and $y_1 = -1$

$y - y_1 = m(x - x_1)$

$y - (-1) = 3(x - 1)$

$y + 1 = 3x - 3$

$y = 3x - 4$

This is the required equation of the tangent.

The normal to a curve is the line perpendicular to the tangent. Remember that the gradient of a line perpendicular to a line with gradient m is m', where $m' = -\frac{1}{m}$.

Example 5

Show that the normal to the curve $y = 2x^2 - 3x$ at the point (1, −1) passes through the origin.

Solution

$y = 2x^2 - 3x \Rightarrow \frac{dy}{dx} = 4x - 3.$

First, find the gradient of the tangent by differentiating y.

At the point with x–coordinate 1, $\frac{dy}{dx} = (4 \times 1) - 3 = 1$

Gradient of the tangent = 1, so gradient of the normal = $-\frac{1}{1} = -1$

$y - y_1 = m(x - x_1)$

$y - (-1) = -1(x - 1)$

$y + 1 = -x + 1$

$y = -x$

This line passes through the origin.

Use the formula for the equation of a line with $m = -1$, $x_1 = 1$ and $y_1 = -1$

For practice in questions like Examples 4 and 5, try the ***Tangents and normal skill pack***.

Example 6

A curve has equation $y = x^3 - x^2 + x + 2$.

(i) Find the x-coordinates of the points on the curve with gradient 6.

(ii) Find the x-coordinates of the points on the curve for which the gradient of the normal is $-\frac{1}{2}$.

Solution

$$y = x^3 - x^2 + x + 2 \Rightarrow \frac{dy}{dx} = 3x^2 - 2x + 1$$

(i) Gradient = 6 $\Rightarrow 3x^2 - 2x + 1 = 6$

$\Rightarrow 3x^2 - 2x - 5 = 0$

$\Rightarrow (3x - 5)(x + 1) = 0$

$\Rightarrow x = \frac{5}{3}$ or $x = -1$

(ii) Gradient of normal = $-\frac{1}{2} \Rightarrow$ gradient of curve = 2

Gradient = 2 $\Rightarrow 3x^2 - 2x + 1 = 2$

$\Rightarrow 3x^2 - 2x - 1 = 0$

$\Rightarrow (3x + 1)(x - 1) = 0$

$\Rightarrow x = -\frac{1}{3}$ or $x = 1$

Section 1: Introduction to differentiation

Exercise level 1

1. Differentiate with respect to x:
   (i) $f(x) = 2x + 1$ (ii) $f(x) = x^3 - 5x$ (iii) $f(x) = x(x+2)$.

2. For the curve $y = 2x^3 - 3x^2 + x$
   (i) Find $\frac{dy}{dx}$
   (ii) Find the gradient of the curve at the point where $x = -2$.

3. Given that $y = 12x - x^3$,
   (i) Find the gradient of the curve at the origin.
   (ii) Find the coordinates of the two points where the gradient is zero.

4. Find the equation of the tangent to the curve $y = x^4 - x + 1$ at the point with x–coordinate 1.

5. (i) Show that the equation of the normal to the curve $y = x^2 - x$ at the point (3, 6) is $x + 5y = 33$.
   (ii) Find the coordinates of the point where the normal meets the x–axis.

AQA AS Mathematics Differentiation

Section 1: Introduction to differentiation

Solutions to Exercise level 1

1. (i) $f(x) = 2x + 1$
   $f'(x) = 2$

   (ii) $f(x) = x^3 - 5x$
   $f'(x) = 3x^2 - 5$

   (iii) $f(x) = x(x+2) = x^2 + 2x$
   $f'(x) = 2x + 2$

2. (i) $y = 2x^3 - 3x^2 + x$
   $\frac{dy}{dx} = 6x^2 - 6x + 1$

   (ii) When $x = -2$, gradient $= 6(-2)^2 - 6(-2) + 1$
   $= 24 + 12 + 1$
   $= 37$

3. (i) $y = 12x - x^3$
   $\frac{dy}{dx} = 12 - 3x^2$
   When $x = 0$, $\frac{dy}{dx} = 12$
   The gradient of the curve at the origin is 12.

   (ii) When gradient is zero, $12 - 3x^2 = 0$
   $4 - x^2 = 0$
   $(2 + x)(2 - x) = 0$
   $x = -2$ or $x = 2$
   When $x = -2$, $y = 12 \times -2 - (-2)^3 = -24 + 8 = -16$
   When $x = 2$, $y = 12 \times 2 - 2^3 = 24 - 8 = 16$
   The gradient is zero at (-2, -16) and (2, 16).

4. $y = x^4 - x + 1$
   $\frac{dy}{dx} = 4x^3 - 1$

When x = 1, $\frac{dy}{dx} = 4 \times 1^3 - 1 = 4 - 1 = 3$

When x = 1, $y = 1^4 - 1 + 1 = 1$

The tangent is the straight line with gradient 3 passing through (1, 1).

Equation of tangent is $y - 1 = 3(x - 1)$

$$y - 1 = 3x - 3$$

$$y = 3x - 2$$

5. $y = x^2 - x$

$\frac{dy}{dx} = 2x - 1$

When x = 3, $\frac{dy}{dx} = 2 \times 3 - 1 = 5$

Gradient of tangent = 5, so gradient of normal $= -\frac{1}{5}$.

The normal is the straight line with gradient $-\frac{1}{5}$ passing through (3, 6).

Equation of normal is $y - 6 = -\frac{1}{5}(x - 3)$

$$5(y - 6) = -(x - 3)$$

$$5y - 30 = -x + 3$$

$$5y + x = 33$$

Where the normal meets the x-axis, y = 0 so x = 33.

The normal meets the x-axis at (33, 0).

Section 1: Introduction to differentiation

Exercise level 2

1. Given that $y = x^3 + 2x^2$, find $\frac{dy}{dx}$. Hence find the x–coordinates of the two points on the curve where the gradient is 4.

2. (i) Show that the point (1, 2) lies on both the curves $y = 2x^3$ and $y = 3x^2 - 1$.
   (ii) Show that the curves have the same gradient at this point.
   (iii) What do these results this tell you about the two curves?

3. The displacement s metres of a particle from a point O after t seconds is given by the equation $s = t^3 - 3t^2 - 9t$. Find the velocity v ($= \frac{ds}{dt}$) in terms of t, and hence find the time at which the particle is stationary (i.e. the velocity is zero).

4. Find $\frac{dy}{dx}$ if:
   (i) $y = (x^2 + 1)(x - 1)$
   (ii) $y = (x - 1)(x + 1)(x - 2)$

5. A curve has equation $y = ax^3 + bx$, where a and b are constants. At the point where $x = 1$, the y–coordinate is 8 and the gradient is 12. Find a and b.

6. Show that the tangent to the curve $y = x^3 + x + 2$ at the point P with x–coordinate 1 passes through the origin, and find the equation of the normal at this point. Given that the normal cuts the x –axis at the point Q, find the area of triangle OPQ.

7. (i) For the graph $y = ax^2 + bx + c$, find the equation of the tangent when $x = p$.
   (ii) Find the equation of the tangent from (i) above, in the case that $b = 0$.
   (iii) Explain by reference to the graph why the answer to (ii) is unchanged for all values of a if $p = 0$.

8. (i) Show that the graphs

   $y = \frac{1}{3}x^3 + 2x + 1$ (A)

   $y = x^2 - \frac{1}{2}x + 1$ (B)

   cross at the point P with coordinates (0, 1).
   (ii) Find the gradients of the two curves at P.
   (iii) What can you deduce about the two curves from your results in (ii) above?
   (iv) Show that for any value of a, the curve $y = ax^2 - \frac{1}{2}x + 1$ crosses the curve (A) above at a constant angle.

Section 1: Introduction to differentiation

Solutions to Exercise level 2

1. $y = x^3 + 2x^2$

$\frac{dy}{dx} = 3x^2 + 4x$

When gradient is 4, $3x^2 + 4x = 4$

$3x^2 + 4x - 4 = 0$

$(3x - 2)(x + 2) = 0$

$x = \frac{2}{3}$ or $x = -2$

2. (i) When $x = 1$, $y = 2x^3 = 2 \times 1^3 = 2$

When $x = 1$, $y = 3x^2 - 1 = 3 \times 1^2 - 1 = 2$

so the point (1, 2) lies on both curves.

(ii) $y = 2x^3$

$\frac{dy}{dx} = 6x^2$

When $x = 1$, gradient $= 6 \times 1 = 6$

$y = 3x^2 - 1$

$\frac{dy}{dx} = 6x$

When $x = 1$, gradient $= 6 \times 1 = 6$

so the curves have the same gradient at this point.

(iii) The two curves touch each other at (1, 2).

3. $s = t^3 - 3t^2 - 9t$

$v = \frac{ds}{dt} = 3t^2 - 6t - 9$

When particle is stationary, $3t^2 - 6t - 9 = 0$

$t^2 - 2t - 3 = 0$

$(t - 3)(t + 1) = 0$

$t = 3$ or $t = -1$

Since time must be positive, $t = 3$.

The particle is stationary after 3 seconds.

4. (i) $y = x^3 - x^2 + x - 1$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 2x + 1$$

(ii) $y = (x^2 - 1)(x - 2)$

$$= x^3 - 2x^2 - x + 2$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 4x - 1$$

5. $y = ax^3 + bx$

When x = 1, $y = a + b \quad \Rightarrow a + b = 8$

$$\frac{dy}{dx} = 3ax^2 + b$$

When x = 1, gradient $= 3a + b \quad \Rightarrow 3a + b = 12$

$$3a + b = 12$$

$$a + b = 8$$

Subtracting: $2a = 4$

$$a = 2, b = 6$$

6. $y = x^3 + x + 2$

$$\frac{dy}{dx} = 3x^2 + 1$$

When x = 1, $\frac{dy}{dx} = 3 \times 1^2 + 1 = 4$

When x = 1, $y = 1^3 + 1 + 2 = 4$

The tangent has gradient 4 and passes through the point (1, 4).

Equation of tangent is $y - 4 = 4(x - 1)$

$$y - 4 = 4x - 4$$

$$y = 4x$$

So the tangent passes through the origin.

Gradient of normal $= -\frac{1}{4}$

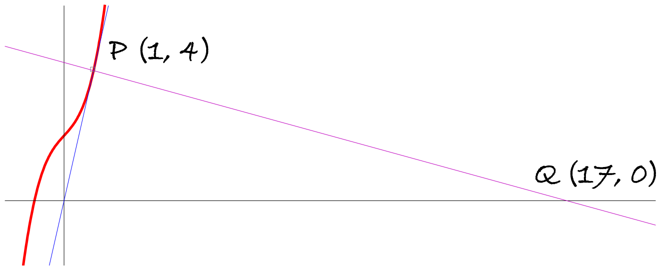
Equation of normal is $y - 4 = -\frac{1}{4}(x - 1)$

$$4(y - 4) = -(x - 1)$$

$$4y - 16 = -x + 1$$

$$4y + x = 17$$

When $y = 0$, $x = 17$, so Q is (17, 0).



Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 17 \times 4 = 34$

7. (i) $x = p \Rightarrow y = ap^2 + bp + c$

$\frac{dy}{dx} = 2ax + b$

so $x = p \Rightarrow \frac{dy}{dx} = 2ap + b$

Equation of tangent is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$

$y = (2ap + b)x - ap^2 + c$

(ii) $y = 2apx - ap^2 + c$

(ii) If $b = 0$, the equation is $y = ax^2 + c$, so $(0, c)$ is the vertex of the graph. So for all values of a, the equation of the tangent at $x = 0$ is always $y = c$.

8. (i) $x = 0$, so curve (A) gives $y = 1$, and curve B gives $y = 1$, so the curves cross at $(0, 1)$.

(ii) For (A), $\frac{dy}{dx} = x^2 + 2$

so $x = 0 \Rightarrow$ gradient of curve = 2.

For (B), $\frac{dy}{dx} = 2x - \frac{1}{2}$,

so $x = 0 \Rightarrow$ gradient of curve $= -\frac{1}{2}$

(iii) The tangents of the two curves are perpendicular – they cross at right-angles.

(iv) For $y = ax^2 - \frac{1}{2}x + 1$, $\frac{dy}{dx} = 2ax - \frac{1}{2}$

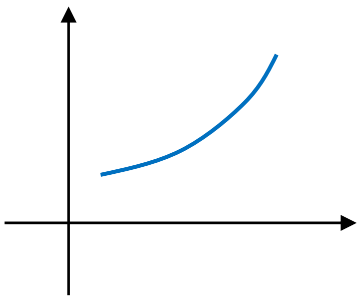
so at $(0, 1)$, gradient of curve $= -\frac{1}{2}$ for any value of a.

Section 2: Maximum and minimum points

These notes contain sub-sections on:

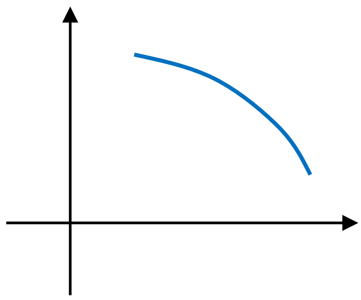
Increasing and decreasing functions

When the gradient $\frac{dy}{dx}$ of a graph is positive, the value of y is increasing.
Similarly, when the gradient is negative, the value of y is decreasing.



$\frac{dy}{dx} > 0$

Increasing function



$\frac{dy}{dx} < 0$

Decreasing function

Example 1

Find the range of values of x for which $y = x^3 - 3x^2 - 9x + 4$ is increasing.

Solution

$y = x^3 - 3x^2 - 9x + 4$

$\frac{dy}{dx} = 3x^2 - 6x - 9$

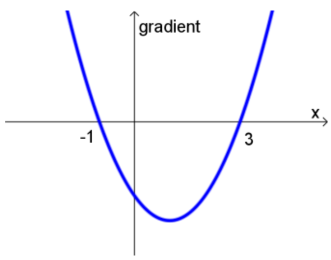
The function is increasing if $\frac{dy}{dx} > 0$

$\Rightarrow 3x^2 - 6x - 9 > 0$

$\Rightarrow x^2 - 2x - 3 > 0$

$\Rightarrow (x-3)(x+1) > 0$

$\Rightarrow x < -1 \text{ or } x > 3$



So the function is increasing for $x < -1$ and for $x > 3$

AQA AS Maths Differentiation 2 Notes and Examples

Turning points

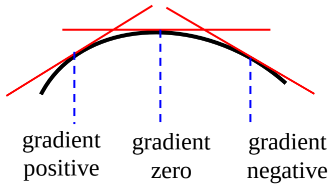
Points on a curve where the tangent is horizontal are called *stationary points*, or *turning points.*

At these points, the gradient of the curve is zero, so $\frac{dy}{dx} = 0$.

You will be looking at two types of stationary point:

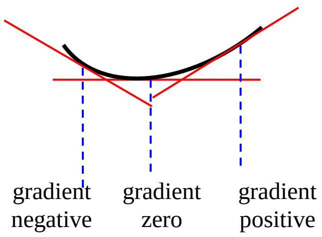
Local **maximum**

The gradient is positive to the left, zero at the point, and negative to the right.



Local **minimum**

The gradient is negative to the left, zero at the point, and positive to the right.



To distinguish between these, you can test the value of the derivative either side of the stationary point, to see whether the gradient is positive or negative.

Example 2

Find the stationary points on the curve $y = x^3 - 3x^2 + 1$, investigate their nature, and sketch the curve.

Solution

$y = x^3 - 3x^2 + 1$

$\frac{dy}{dx} = 3x^2 - 6x$

Step 1: Differentiate the function.

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

Step 2: Solve $\frac{dy}{dx} = 0$

When $x = 0$, $y = 0^3 - 3\times 0^2 + 1 = 1$

When $x = 2$, $y = 2^3 - 3\times 2^2 + 1 = 8 - 12 + 1 = -3$.

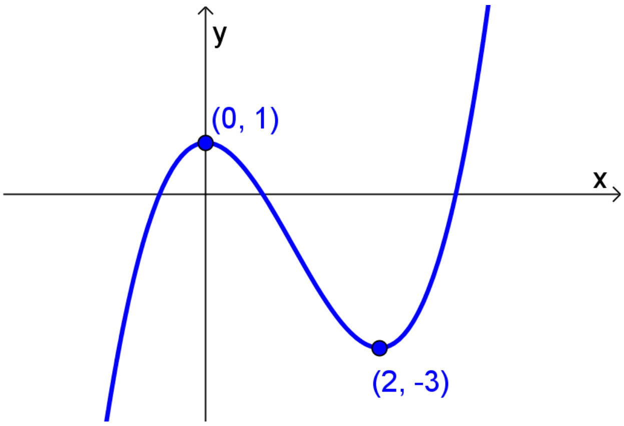
Step 3: Calculate the *y*–coordinates for these values of *x* (called the stationary values).

So the stationary points are (0, 1) and (2, –3).

Step 4: Use a table to investigate the sign of $\frac{dy}{dx}$ for values of x either side of the stationary values

x	-1	0	1	2	2
$\frac{dy}{dx}$	9 +ve	0	-3 -ve	0	9 +ve

So (0, 1) is a local maximum and (2, -3) is a local minimum



Step 5: Sketch the curve.

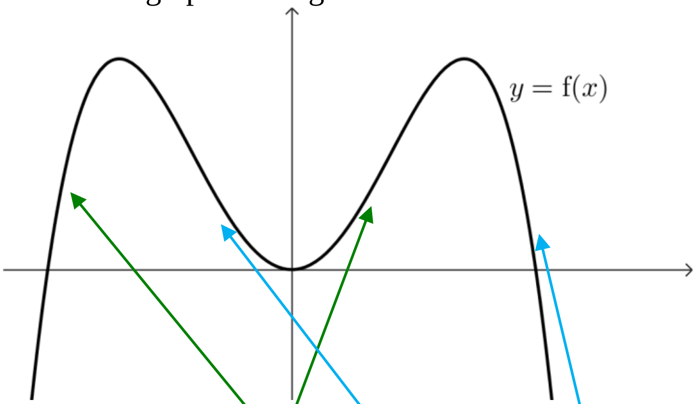
You can investigate stationary points on quadratic and cubic curves using the Explore resource ***Stationary points***. Also try the ***Stationary points walkthrough***.

Sketching the graph of a derivative

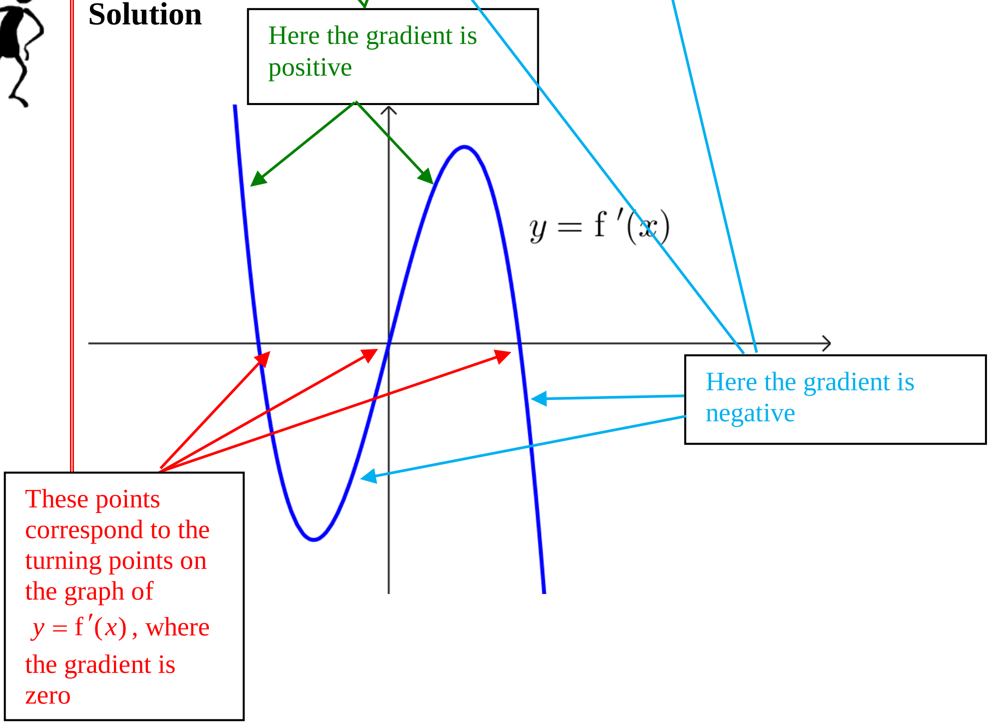
If you have the graph of a function $y = f(x)$, you can sketch the graph of the corresponding gradient function, $y = f'(x)$, by thinking about what is happening at different points on the graph.

- Where there is a turning point, the gradient of $y = f(x)$ is zero so the gradient graph $y = f'(x)$ crosses the x-axis
- Where the graph is increasing, the gradient of $y = f(x)$ is positive so the gradient graph will be above the x-axis
- Where the graph is decreasing, the gradient of $y = f(x)$ is negative so the gradient graph will be below the x-axis

Example 3
Sketch the graph of the gradient function of the curve shown below.



Solution



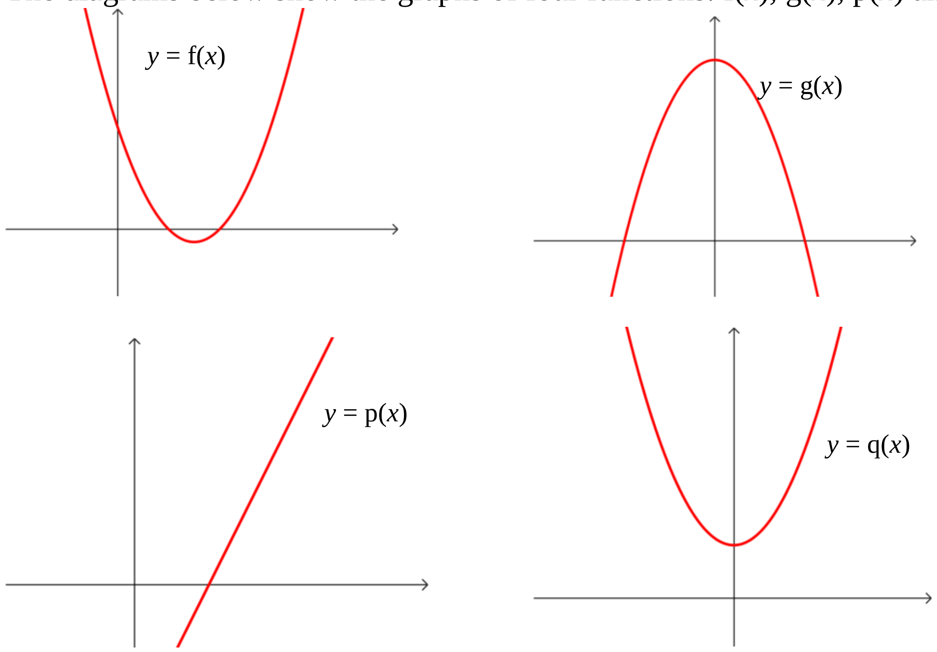
Section 2: Maximum and minimum points

Exercise level 1

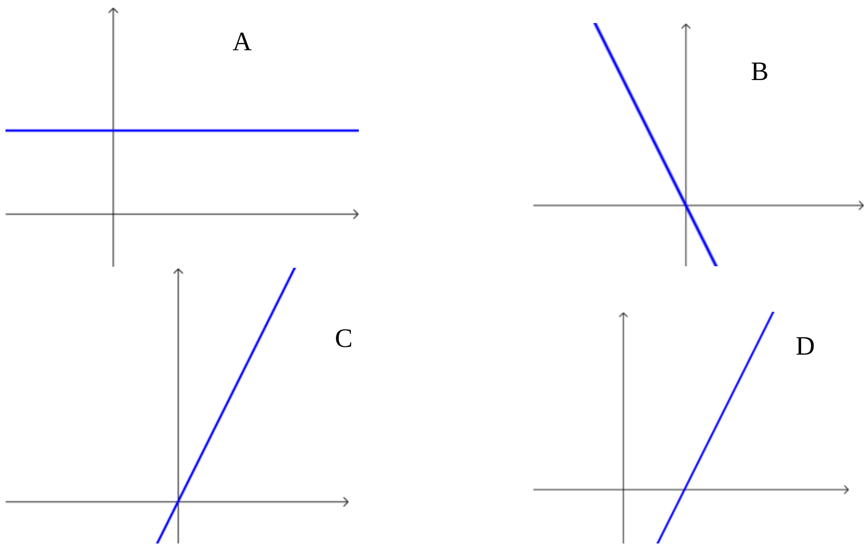
1. Find the range of values of x for which $f(x) = 2x^2 - 3x + 1$ is an increasing function.

2. Find the range of values of x for which $f(x) = 4 + 7x - 3x^2$ is a decreasing function.

3. The diagrams below show the graphs of four functions: f(x), g(x), p(x) and q(x).



The diagrams below show the gradient functions of f(x), g(x), p(x) and q(x). Match the diagrams A, B, C and D to the equations $y = f'(x)$, $y = g'(x)$, $y = p'(x)$ and $y = q'(x)$.



MEI AS Maths Differentiation 2 Exercise

4. A curve has equation $y = x^3 + 6x^2 + 9x$.

   (i) Differentiate the function to obtain $\frac{dy}{dx}$.

   (ii) Find the x coordinates of the points where $\frac{dy}{dx} = 0$ and hence the coordinates of the turning points on the curve.

   (iii) By considering the sign of $\frac{dy}{dx}$ on either side of the turning points, determine whether the turning points are maximum or minimum points.

   (iv) Sketch the curve showing the turning points and points of intersection with the axes clearly.

Section 2: Maximum and minimum points

Solutions to Exercise level 1

1. $f(x) = 2x^2 - 3x + 1$

   $f'(x) = 4x - 3$

   When $f(x)$ is increasing, $f'(x) > 0$

   $\Rightarrow 4x - 3 > 0$

   $\Rightarrow 4x > 3$

   $\Rightarrow x > \frac{3}{4}$

2. $f(x) = 4 + 7x - 3x^2$

   $f'(x) = 7 - 6x$

   When $f(x)$ is decreasing, $f'(x) < 0$

   $\Rightarrow 7 - 6x < 0$

   $\Rightarrow 7 < 6x$

   $\Rightarrow 6x > 7$

   $\Rightarrow x > \frac{7}{6}$

3. The gradient of f(x) starts as negative, becomes zero and then becomes positive. This could be either C or D, but in C the gradient is zero when $x = 0$, so it must be D.

   The gradient of g(x) starts as positive, is zero when $x = 0$ and then becomes positive. This is graph B.

   The gradient of p(x) is a constant positive value. This is graph A.

   The gradient of q(x) starts as negative, becomes zero when $x = 0$, and then becomes positive. This is graph C.

4. (i) $y = x^3 + 6x^2 + 9x$

   $\frac{dy}{dx} = 3x^2 + 12x + 9$

(ii) $\frac{dy}{dx} = 0$

$3x^2 + 12x + 9 = 0$

$x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$x = -1$ or $x = -3$

When x = -1, $y = (-1)^3 + 6(-1)^2 + 9 \times -1 = -1 + 6 - 9 = -4$

When x = -3, $y = (-3)^3 + 6(-3)^2 + 9 \times -3 = -27 + 54 - 27 = 0$

The turning points are (-1, -4) and (-3, 0)

(iii)

x	x< -3	x = -3	-3 < x < -1	x = -1	x > -1
$\frac{dy}{dx}$	+ve /	0 —	-ve \	0 —	+ve /

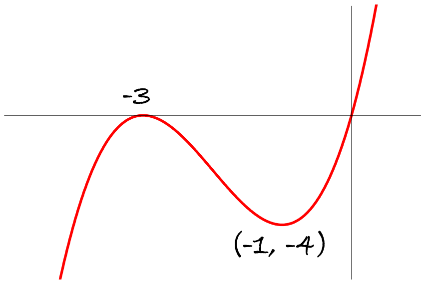
The point (-3, 0) is a maximum point.
The point (-1, -4) is a minimum point.

(iv) $y = x^3 + 6x^2 + 9x$

$= x(x^2 + 6x + 9)$

$= x(x+3)^2$

The graph cuts the x-axis at x = 0 and x = -3 (repeated).
The graph cuts the y-axis at y = 0.



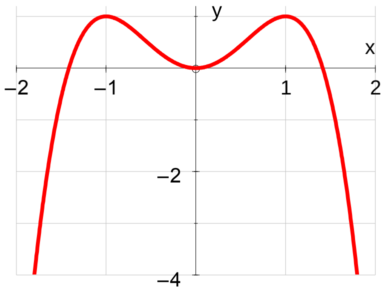
Section 2: Maximum and minimum points

Exercise level 2

1. Find the range of values of x for which $f(x) = x^3 + x^2 - x + 3$ is an increasing function.

2. Find the range of values of x for which $f(x) = x^3 - 6x^2 + 9x + 5$ is a decreasing function.

3. Copy the curve shown below, and sketch the shape of the derivative on the same axes.



4. The equation of a curve is given by $y = 2x + x^2 - 4x^3$.
   (i) Find the coordinates of the turning points on the curve, and distinguish between them by considering the gradient on either side of the turning points.
   (ii) Sketch the curve marking the turning points and points of intersection with the axes clearly.

5. The curve $y = x^3 + px^2 + q$ has a minimum point at (4, -11). Find the coordinates of the maximum point on the curve.

6. The curve $y = x^3 + ax^2 + bx + c$ passes through the point (1, 1).
   (i) Write down and simplify an equation connecting a, b and c.

   The curve also has turning points when $x = -1$ and when $x = 3$.

   (ii) Find two further equations connecting a, b and c.
   (iii) Solve the three equations simultaneously to obtain values for a, b and c.

Section 2: Maximum and minimum points

Solutions to Exercise level 2

1. $f(x) = x^3 + x^2 - x + 3$

$f'(x) = 3x^2 + 2x - 1$

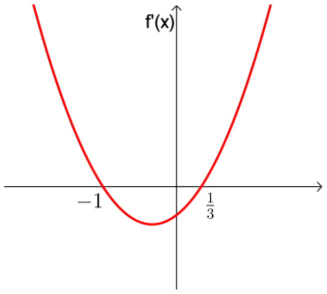
When f(x) is an increasing function, $f'(x) > 0$

$\Rightarrow 3x^2 + 2x - 1 > 0$

$\Rightarrow (3x - 1)(x + 1) > 0$

$\Rightarrow x > \frac{1}{3}$ or $x < -1$

So f(x) is increasing for $x < -1$ and $x > \frac{1}{3}$.



2. $f(x) = x^3 - 6x^2 + 9x + 5$

$f'(x) = 3x^2 - 12x + 9$

When f(x) is a decreasing function, $f'(x) < 0$

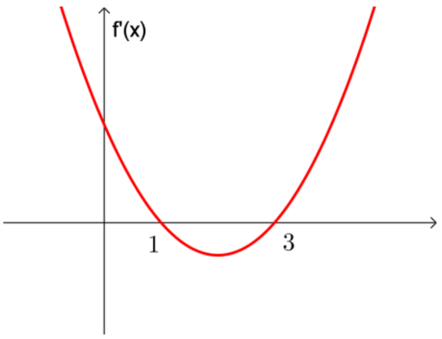
$\Rightarrow 3x^2 - 12x + 9 < 0$

$\Rightarrow x^2 - 4x + 3 < 0$

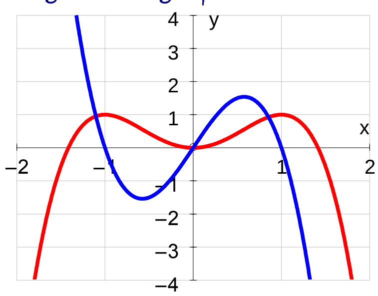
$\Rightarrow (x - 1)(x - 3) > 0$

$\Rightarrow 1 < x < 3$

So f(x) is decreasing for $1 < x < 3$.



3. gradient graph



4. (i) $y = 2x + x^2 - 4x^3$

$\frac{dy}{dx} = 2 + 2x - 12x^2$

At turning points, $\frac{dy}{dx} = 0$

$2 + 2x - 12x^2 = 0$

$1 + x - 6x^2 = 0$

$6x^2 - x - 1 = 0$

$(3x+1)(2x-1) = 0$

$x = -\frac{1}{3}$ or $x = \frac{1}{2}$

When $x = -\frac{1}{3}$, $y = 2\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 - 4\left(-\frac{1}{3}\right)^3 = -\frac{2}{3} + \frac{1}{9} + \frac{4}{27} = \frac{-18+3+4}{27} = -\frac{11}{27}$

When $x = \frac{1}{2}$, $y = 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)^3 = 1 + \frac{1}{4} - \frac{1}{2} = \frac{3}{4}$

The turning points are $\left(-\frac{1}{3}, -\frac{11}{27}\right)$ and $\left(\frac{1}{2}, \frac{3}{4}\right)$.

x	$x < -\frac{1}{3}$	$x = -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$\frac{dy}{dx}$	-ve ╲	0 —	+ve ╱	0 —	-ve ╲

$\left(-\frac{1}{3}, -\frac{11}{27}\right)$ is a minimum point.

$\left(\frac{1}{2}, \frac{3}{4}\right)$ is a maximum point.

(ii) $y = 2x + x^2 - 4x^3$

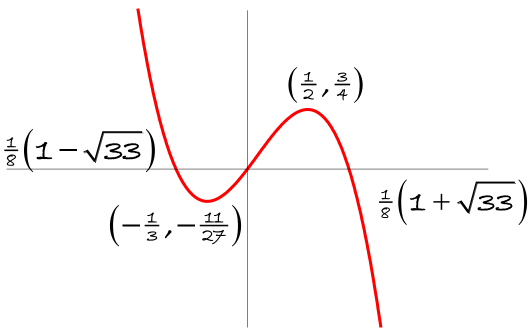
$= x(2 + x - 4x^2)$

$= -x(4x^2 - x - 2)$

The curve cuts the x-axis at $x = 0$ and at the points satisfying $4x^2 - x - 2 = 0$.

For this quadratic equation, $a = 4, b = -1, c = -2$

Using the quadratic formula, $x = \frac{1 \pm \sqrt{1 - 4 \times 4 \times -2}}{8} = \frac{1 \pm \sqrt{33}}{8}$



AQA AS Maths Differentiation 2 Exercise solutions

5. $y = x^3 + px^2 + q$

$\frac{dy}{dx} = 3x^2 + 2px$

At turning points, $\frac{dy}{dx} = 0$

$3x^2 + 2px = 0$

$x(3x + 2p) = 0$

$x = 0$ or $x = -\frac{2p}{3}$

Since there is a minimum point at $x = 4$, $-\frac{2p}{3} = 4 \quad \Rightarrow p = -6$

The curve is therefore $y = x^3 - 6x^2 + q$.

The point (4, -11) lies on the curve, so $-11 = 4^3 - 6 \times 4^2 + q$

$-11 = 64 - 96 + q$

$q = 21$

The equation of the curve is $y = x^3 - 6x^2 + 21$.

The other turning point is at $x = 0$, so the maximum point is (0, 21).

6. (i) $y = x^3 + ax^2 + bx + c$

The graph passes through the point (1, 1)

so $1 = 1 + a + b + c$

$a + b + c = 0$

(ii) $\frac{dy}{dx} = 3x^2 + 2ax + b$

Turning points are when $3x^2 + 2ax + b = 0$

There is a turning point when $x = -1$, so $3(-1)^2 + 2a \times -1 + b = 0$

$3 - 2a + b = 0$

$2a - b = 3$

There is a turning point when $x = 3$, so $3 \times 3^2 + 2a \times 3 + b = 0$

$27 + 6a + b = 0$

$6a + b = -27$

(iii) $a + b + c = 0$ (1)

$2a - b = 3$ (2)

$6a + b = -27$ (3)

Adding (2) and (3): $8a = -24 \quad \Rightarrow a = -3$

Substituting into (2) gives: $b = 2a - 3 = -6 - 3 = -9$

Substituting into (1) gives: $c = -a - b = 9 + 3 = 12$

$a = -3, b = -9, c = 12$

Section 3: Extending the rule

Notes and Examples

These notes contain subsections on

- Differentiating kx^n for negative and fractional n
- Applications of differentiation

Differentiating kx^n for negative and fractional n

You already know that the derivative, or gradient of x^n, where n is a positive integer, is given by nx^{n-1}.

In fact this formula for the derivative of x^n is true not only when n is a positive integer, but for all real values of n, including negative numbers and fractions.

Example 1

Differentiate the following functions.

(i) $y = \dfrac{1}{x}$ (ii) $y = x^2\sqrt{x}$ (iii) $y = \dfrac{1}{\sqrt{x}}$

Solution

(i) $y = \dfrac{1}{x} = x^{-1}$

$\dfrac{dy}{dx} = -1x^{-2} = -\dfrac{1}{x^2}$

Subtracting 1 from -1 gives -2

(ii) $y = x^2\sqrt{x} = x^2x^{\frac{1}{2}} = x^{\frac{5}{2}}$

Use the laws of indices to express this as a single power of x

$\dfrac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$

Subtracting 1 from $\frac{5}{2}$ gives $\frac{3}{2}$

(iii) $y = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$

Subtracting 1 from $-\frac{1}{2}$ gives $-\frac{3}{2}$

For further examples and practice, use the ***Differentiating rational powers of x skill pack***.

AQA AS Maths Differentiation 3 Notes and Examples

You can extend this idea to allow you to differentiate all functions of the form kx^n, where k is a constant, and sums and differences of such functions.

- The derivative of kx^n is knx^{n-1}, where k is a constant and n is any real number
- The derivative of sum (or difference) of two or more such functions is the sum (or difference) of the derivatives of the functions.

Example 2
Differentiate the following functions

(i) $y = (3 - 2x - x^2)\sqrt{x}$

(ii) $y = \dfrac{3x - x^2}{x^5}$

Solution

(i)
$$y = (3 - 2x - x^2)\sqrt{x}$$
$$= 3\sqrt{x} - 2x\sqrt{x} - x^2\sqrt{x}$$
$$= 3x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^{\frac{5}{2}}$$
$$\frac{dy}{dx} = 3 \times \tfrac{1}{2}x^{-\frac{1}{2}} - 2 \times \tfrac{3}{2}x^{\frac{1}{2}} - \tfrac{5}{2}x^{\frac{3}{2}}$$
$$= \tfrac{3}{2}x^{-\frac{1}{2}} - 3x^{\frac{1}{2}} - \tfrac{5}{2}x^{\frac{3}{2}}$$

(ii)
$$y = \frac{3x - x^2}{x^5}$$
$$= \frac{3}{x^4} - \frac{1}{x^3}$$
$$= 3x^{-4} - x^{-3}$$
$$\frac{dy}{dx} = 3 \times -4x^{-5} - (-3x^{-4})$$
$$= -12x^{-5} + 3x^{-4}$$

Applications of differentiation

Now that you can differentiate a wider range of functions, you can also make use of various applications of differentiation in many more contexts. You already know how to use differentiation to find gradients of curves, find the equations of tangents and normals to curves and find maximum and minimum points on curves. The following examples cover these applications.

Example 3
For the graph $y = x - \sqrt{x}$

(i) find the gradient at the point (4, 2)
(ii) find the equation of the tangent at this point
(iii) find the equation of the normal at this point.

Solution

(i) $y = x - \sqrt{x} = x - x^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 - \frac{1}{2}x^{-\frac{1}{2}} = 1 - \frac{1}{2\sqrt{x}}$

When $x = 4$, $\frac{dy}{dx} = 1 - \frac{1}{2\sqrt{4}} = 1 - \frac{1}{4} = \frac{3}{4}$

The gradient at (4, 2) is $\frac{3}{4}$.

(ii) Gradient of tangent at (4, 2) = $\frac{3}{4}$

Equation of tangent at (4, 2) is $y - 2 = \frac{3}{4}(x - 4)$

$y = \frac{3}{4}x - 3 + 2$

$y = \frac{3}{4}x - 1$

> Using the equation of a line $y - y_1 = m(x - x_1)$ with $m = \frac{3}{4}$ and $(x_1, y_1) = (4, 2)$

> Remember that when two lines with gradients m_1 and m_2 are perpendicular, $m_1 m_2 = -1$

(iii) Gradient of normal at (4, 2) = $-\frac{4}{3}$

Equation of normal at (4, 2) is $y - 2 = -\frac{4}{3}(x - 4)$

$y = -\frac{4}{3}x + \frac{16}{3} + 2$

$y = -\frac{4}{3}x + \frac{22}{3}$

> Using the equation of a line $y - y_1 = m(x - x_1)$ with $m = -\frac{4}{3}$ and $(x_1, y_1) = (4, 2)$

Example 4

Find the stationary points of the graph $y = \frac{x^2 - 3}{x^3}$ and determine their nature.

Solution

$y = \frac{x^2 - 3}{x^3} = \frac{1}{x} - \frac{3}{x^3} = x^{-1} - 3x^{-3}$

$\frac{dy}{dx} = -x^{-2} + 9x^{-4} = -\frac{1}{x^2} + \frac{9}{x^4}$

> At stationary points, the gradient is zero. Differentiate the function and find the values of x for which $\frac{dy}{dx} = 0$

At stationary points, $\frac{dy}{dx} = 0$

$-\frac{1}{x^2} + \frac{9}{x^4} = 0 \Rightarrow -x^2 + 9 = 0$

$\Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$

When $x = 3$, $y = \frac{9-3}{27} = \frac{6}{27} = \frac{2}{9}$

When $x = -3$, $y = \frac{9-3}{-27} = -\frac{6}{27} = -\frac{2}{9}$

> Substitute the values of x into the original equation to find the y-coordinates of the stationary points.

The stationary points are $(3, \frac{2}{9})$ and $(-3, -\frac{2}{9})$.

When $x = 2$, gradient is +ve and when $x = 4$, gradient is ï ve
so $\left(3, \frac{2}{9}\right)$ is a local maximum
When $x = -4$, gradient is ï ve and when x = -2, gradient is +ve
so $\left(-3, -\frac{2}{9}\right)$ is a local minimum.

Section 3: Extending the rule

Exercise level 1

1. Differentiate the following functions

(i) $y = \frac{1}{x^3}$ (ii) $y = \sqrt[3]{x}$

(iii) $y = \frac{2}{x} - \frac{3}{x^2}$ (iv) $y = 4\sqrt{x} - \frac{3}{\sqrt{x}}$

(v) $y = 3x^{-5} - 2x^{-7}$ (vi) $y = 2x^{\frac{2}{3}} - 5x^{-\frac{2}{3}}$

(vii) $y = \frac{x^2 - 2x + 3}{2x^2}$ (viii) $y = \left(x^2 - 2\right)\sqrt{x}$

2. Find the gradient of each of the following graphs at the given point

(i) $y = 2x - \frac{1}{x}$ at the point (1, 1)

(ii) $y = 3 - \sqrt{x}$ at the point (4, 1)

(iii) $y = x^2\sqrt{x}$ at the point (1, 1)

3. Find the equation of the tangent to the graph $y = \frac{1}{\sqrt{x}}$ at the point where $x = 1$.

4. Find the equation of the normal to the graph $y = \frac{1}{x} - \frac{2}{x^2}$ at the point where $x = 2$.

Section 3: Extending the rule

Solutions to Exercise level 1

1. (i) $y = \frac{1}{x^3} = x^{-3}$

$\frac{dy}{dx} = -3x^{-4} = -\frac{3}{x^4}$

(ii) $y = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

(iii) $y = \frac{2}{x} - \frac{3}{x^2} = 2x^{-1} - 3x^{-2}$

$\frac{dy}{dx} = -2x^{-2} + 6x^{-3} = -\frac{2}{x^2} + \frac{6}{x^3}$

(iv) $y = 4\sqrt{x} - \frac{3}{\sqrt{x}} = 4x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 2x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$

(v) $y = 3x^{-5} - 2x^{-7}$

$\frac{dy}{dx} = -15x^{-6} + 14x^{-8} = -\frac{15}{x^6} + \frac{14}{x^8}$

(vi) $y = 2x^{\frac{2}{3}} - 5x^{-\frac{2}{3}}$

$\frac{dy}{dx} = \frac{4}{3}x^{-\frac{1}{3}} + \frac{10}{3}x^{-\frac{5}{3}}$

(vii) $y = \frac{x^2 - 2x + 3}{2x^2} = \frac{1}{2} - x^{-1} + \frac{3}{2}x^{-2}$

$\frac{dy}{dx} = x^{-2} - 3x^{-3} = \frac{1}{x^2} - \frac{3}{x^3}$

(viii) $y = (x^2 - 2)(\sqrt{x}) = x^{\frac{5}{2}} - 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} - x^{-\frac{1}{2}}$

AQA AS Maths Differentiation 3 Exercise solutions

2. (i) $y = 2x - \frac{1}{x} = 2x - x^{-1}$

$$\frac{dy}{dx} = 2 + x^{-2}$$

$$\text{At } (1,\ 1): \frac{dy}{dx} = 2 + (1)^{-2} = 3$$

(ii) $y = 3 - \sqrt{x} = 3 - x^{\frac{1}{2}}$

$$\frac{dy}{dx} = -\tfrac{1}{2}x^{-\frac{1}{2}} = -\frac{1}{2\sqrt{x}}$$

$$\text{At } (4,\ 1): \frac{dy}{dx} = -\frac{1}{2\sqrt{4}} = -\frac{1}{4}$$

(iii) $y = x^2\sqrt{x} = x^{\frac{5}{2}}$

$$\frac{dy}{dx} = \tfrac{5}{2}x^{\frac{3}{2}}$$

$$\text{At } (1,\ 1): \frac{dy}{dx} = \tfrac{5}{2}(1)^{\frac{3}{2}} = \tfrac{5}{2}$$

3. $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \quad \Rightarrow \frac{dy}{dx} = -\tfrac{1}{2}x^{-\frac{3}{2}}$

When $x = 1, y = 1$ and $\frac{dy}{dx} = -\tfrac{1}{2}(1)^{-\frac{3}{2}} = -\tfrac{1}{2}$

$$y - y_1 = m(x - x_1)$$
$$y - 1 = -\tfrac{1}{2}(x - 1)$$
$$y - 1 = -\tfrac{1}{2}x + \tfrac{1}{2}$$
$$y = -\tfrac{1}{2}x + \tfrac{3}{2}$$

4. $y = \frac{1}{x} - \frac{2}{x^2} = x^{-1} - 2x^{-2} \quad \Rightarrow \frac{dy}{dx} = -x^{-2} + 4x^{-3}$

When $x = 2, y = \tfrac{1}{2} - \tfrac{1}{2} = 0, \frac{dy}{dx} = -(2)^{-2} + 4(2)^{-3} = 0.25$

As normal, $m = \frac{-1}{0.25} = -4$

$$y - y_1 = m(x - x_1)$$
$$y - 0 = -4(x - 2)$$
$$y = -4x + 8$$

Section 3: Extending the rule

Exercise level 2

1. Differentiate the following expressions with respect to x:
   (i) $y = x^{-\frac{1}{2}}$
   (ii) $y = -\dfrac{1}{x^4}$
   (iii) $y = (x-2)^2\sqrt{x}$
   (iv) $y = \dfrac{(x+1)^2}{\sqrt{x}}$

2. Find the coordinates of the point P where the graph of $y = x^2 + \dfrac{1}{x}$ crosses the x-axis, and hence find the equations of the tangent and normal through the point P.

3. Find any stationary points on the following curves and determine their nature.
   (i) $y = x - \dfrac{4}{x^2}$
   (ii) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

4. The point P with x-coordinate 1 lies on the curve $y = \dfrac{2}{x}$.
   Q is the point where the normal at P meets the curve again.
   R is the point where the tangents at P and Q meet.
   Find the area of triangle PQR. Give your answer in exact form.

Section 3: Extending the rule

Solutions to Exercise level 2

1. (i) $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$

$= -\frac{1}{2x^{\frac{3}{2}}}$

(ii) $\frac{dy}{dx} = -(-4)x^{-5}$

$= \frac{4}{x^5}$

(iii) $y = (x^2 - 4x + 4)\sqrt{x}$

$= x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$

$= \frac{1}{2}x^{\frac{1}{2}}\left(5x - 12 + \frac{2}{x}\right)$

(iv) $y = \frac{x^2 + 2x + 1}{\sqrt{x}}$

$= x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{1}{2}x^{\frac{1}{2}}\left(3 + \frac{2}{x} - \frac{1}{x^2}\right)$

2. When $y = 0$, $x^2 + \frac{1}{x} = 0 \Rightarrow x^3 = -1 \Rightarrow x = -1$

so P = (-1, 0).

$\frac{dy}{dx} = 2x - \frac{1}{x^2}$

so when $x = -1$, $\frac{dy}{dx} = -3$

Equation of tangent through (-1, 0) is $y - 0 = -3(x - (-1))$

$\Rightarrow y = -3x - 3$

Gradient of normal at (-1, 0) is $\frac{1}{3}$

Equation of normal through (-1, 0) is $y - 0 = \frac{1}{3}(x - (-1))$

$\Rightarrow 3y = x + 1$

3. (i) $y = x - \frac{4}{x^2} = x - 4x^{-2}$

$\frac{dy}{dx} = 1 + 8x^{-3}$

Stationary points occur when $\frac{dy}{dx} = 0$.

$0 = 1 + 8x^{-3}$

$-1 = 8x^{-3}$

$x^3 = -8$

$x = -2$

When $x = -2$, $y = -3$

$\frac{d^2y}{dx^2} = -24x^{-4}$

At $x = -2, \frac{d^2y}{dx^2} = -24(-2)^{-4} = -1.5$

As this is negative, $(-2, -3)$ is a local maximum.

(ii) $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

Stationary points occur when $\frac{dy}{dx} = 0$.

$0 = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{3}{2}}$

$x^{-\frac{1}{2}} = x^{-\frac{3}{2}}$

$1 = x^{-1}$

$x = 1$

When $x = 1$, $y = 2$

$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

At $x = 1$, $\frac{d^2y}{dx^2} = -\frac{1}{4} + \frac{3}{4} = \frac{1}{2}$

As this is positive, $(1, 2)$ is a local minimum.

AQA AS Maths Differentiation 3 Exercise solutions

4. The coordinates of P are (1, 2).

$$y = \frac{2}{x} \quad \Rightarrow \frac{dy}{dx} = -\frac{2}{x^2}$$

At P, gradient of tangent $= -\frac{2}{1^2} = -2$ and so gradient of normal $= -\frac{1}{2}$.

Equation of normal at P is $y - 2 = \frac{1}{2}(x - 1)$

$$y = \frac{1}{2}x + \frac{3}{2}$$

At Q, this normal meets the curve again, so $\frac{1}{2}x + \frac{3}{2} = \frac{2}{x}$

$$x^2 + 3x = 4$$

$$x^2 + 3x - 4 = 0$$

$$(x - 1)(x + 4) = 0$$

so $x = 1$ (which is P) or $x = -4$ (which is Q).

Coordinates of Q are $(-4, -\frac{1}{2})$

Gradient of tangent at P is -2 so equation of tangent is $y - 2 = -2(x - 1)$

$$y = -2x + 4$$

Gradient of tangent at Q is $-\frac{1}{8}$ so equation of tangent is $y + \frac{1}{2} = -\frac{1}{8}(x + 4)$

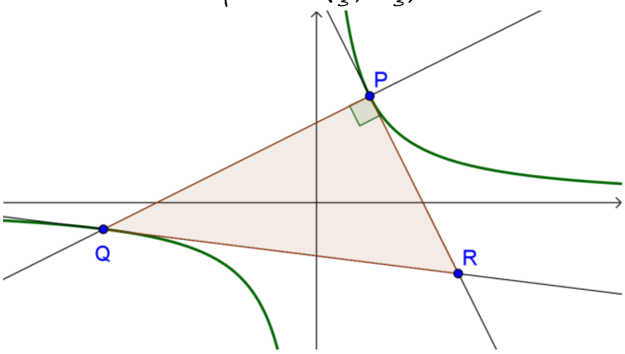
$$y = -\frac{1}{8}x - 1$$

R is where P and Q intersect: $-2x + 4 = -\frac{1}{8}x - 1$

$$5 = \frac{15}{8}x$$

$$x = \frac{8}{3}, y = -\frac{4}{3}$$

So coordinates of R are $(\frac{8}{3}, -\frac{4}{3})$



Since PQ is the normal at P and PR is the tangent at P, the triangle is right-angled .

$$PQ = \sqrt{(1+4)^2 + (2+\frac{1}{2})^2} = \sqrt{25 + \frac{25}{4}} = \sqrt{\frac{125}{4}} = \frac{5}{2}\sqrt{5}$$

$$PR = \sqrt{(1-\frac{8}{3})^2 + (2+\frac{4}{3})^2} = \sqrt{\frac{25}{9} + \frac{100}{9}} = \sqrt{\frac{125}{9}} = \frac{5}{3}\sqrt{5}$$

Area of triangle $= \frac{1}{2} \times \frac{5}{2}\sqrt{5} \times \frac{5}{3}\sqrt{5}$

$$= \frac{125}{12}$$

Section 4: More about differentiation

These notes contain sub-sections on:

- Second derivatives
- The second derivative test for turning points
- Maximum and minimum problems
- Differentiation from first principles

Second derivatives

If you differentiate a derivative, you get the *second derivative*. If you start with an equation for y in terms of x, the first derivative is $\frac{dy}{dx}$ (you say: "dee y by dee x") and the second derivative is written $\frac{d^2y}{dx^2}$ (you say: "dee two y by dee x squared")

The second derivative tells you about the rate of change of the derivative.

Example 1

Given that $y = 3x - x^3$, find $\frac{d^2y}{dx^2}$.

Solution

$y = 3x - x^3$

$\Rightarrow \frac{dy}{dx} = 3 - 3x^2$

$\Rightarrow \frac{d^2y}{dx^2} = -6x$

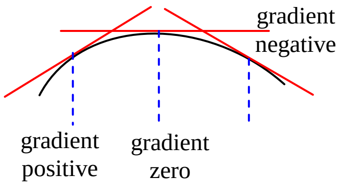
One important application of first and second derivatives is in the motion of a particle. You will learn more about this in your study of Mechanics.

The second derivative test for turning points

Maximum points

If $\frac{d^2y}{dx^2} < 0$, the gradient function $\frac{dy}{dx}$ is decreasing.

At a maximum point, the gradient goes from + to 0 to −, in other words is decreasing.

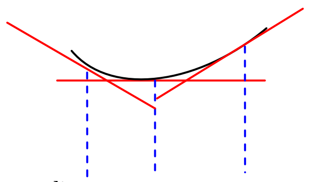


So $\frac{d^2y}{dx^2} < 0$ Ÿ the turning point is a maximum.

Minimum points

If $\frac{d^2y}{dx^2} > 0$, the gradient function $\frac{dy}{dx}$ is increasing.

At a minimum point, the gradient goes from   to 0 to +, in other words is increasing.



So $\frac{d^2y}{dx^2} > 0$ Ÿ the turning point is a minimum.

If the value of the second derivative is zero, this method cannot be used, and you must use the earlier method of looking at the sign of the gradient on either side of the point.

Example 2

Find the turning points of $y \quad 3x \quad x^3$ and determine their nature using the second derivative test.

Solution

$y \quad 3x \quad x^3$

Ÿ $\frac{dy}{dx} \quad 3 \quad 3x^2$

$\frac{dy}{dx}$ 0 when $3 \quad 3x^2 \quad 0$

$1 \quad x^2 \quad 0$

$(1 \quad x)(1 \quad x) \quad 0$

Ÿ $x = 1$ or   1

When $x = 1$, $y = 2$; when $x =$ 1, $y =$ 2

The stationary points are (1, 2) and (1,   2)

$\frac{d^2y}{dx^2} \quad 6x$

When $x = 1$, $\frac{d^2y}{dx^2} \quad 6 \quad 0$ Ÿ maximum

When $x =$ 1, $\frac{d^2y}{dx^2} \quad 6 \quad 0$ Ÿ minimum.

(1, 2) is a maximum point and (-1, -2) is a minimum point.

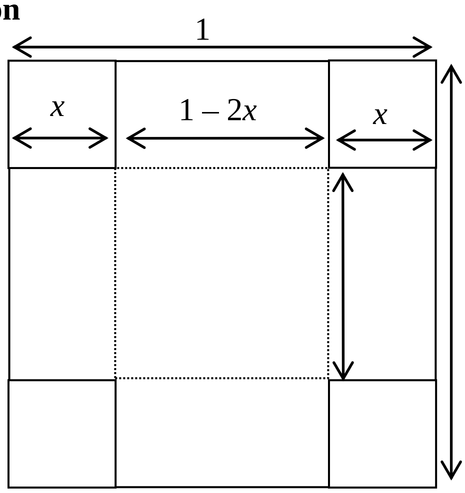
Maximum and minimum problems

One important immediate application of differentiation is to problems that involve maximising or minimising a variable quantity. You have already met the idea of finding maximum or minimum points on a graph using differentiation: now you will start to apply the same ideas to other types of problem.

Example 3

A rectangular sheet of metal of length 1 m and width 1 m has squares cut from each corner. The sides are then folded up to form an open topped box. Find the maximum possible volume of the box.

Solution



Step 1: Draw a diagram and use it to help you to formulate the problem mathematically. Call the side length of the squares cut out x. What are the length, width and depth of the box in terms of x?

Length = width $\quad 1 \quad 2x$

Depth = x

The volume V m^3 of the box is given by $V \quad x(1 \quad 2x)^2$.

$$V \quad x(1 \quad 2x)^2$$
$$x(1 \quad 4x \quad 4x^2)$$
$$x \quad 4x^2 \quad 4x^3$$

Step 2: Find the maximum volume by differentiating. The maximum volume will occur when $\frac{dV}{dx} \quad 0$. Before differentiating, expand the brackets.

$$\frac{dV}{dx} \quad 1 \quad 8x \quad 12x^2$$

$$1 \quad 8x \quad 12x^2 \quad 0$$
$$(2x \quad 1)(6x \quad 1) \quad 0$$
$$x \quad \tfrac{1}{2} \text{ or } x \quad \tfrac{1}{6}$$

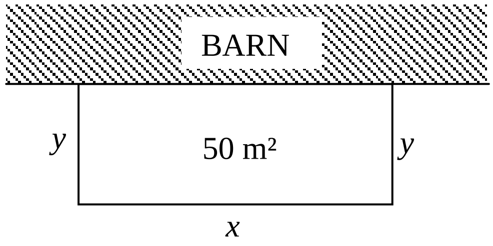
Now put $\frac{dV}{dx} \quad 0$, and solve for x:

When $x \quad \frac{1}{2}, V \quad \frac{1}{2}\left(1 \quad 2 \quad \frac{1}{2}\right)^2 \quad 0$. This must be the minimum.

When $x \quad \frac{1}{6}, V \quad \frac{1}{6}\left(1 \quad 2 \quad \frac{1}{6}\right)^2 \quad \frac{4}{54} \quad \frac{2}{27}$. This must be the maximum.

So the maximum possible volume of the box is $\frac{2}{27}$ m^3.

Example 4
A farmer wants to make a pen for a goat, using the side of a barn as one side of the pen. He wants the pen to have an area of 50 m², but wants to use as little fencing as possible.



(i) Write down expressions for the area of the pen and the length of fencing required in terms of x and y.
(ii) Find an expression for the length of fencing required, L, in terms of x only.
(iii) Find $\frac{dL}{dx}$ and hence find the minimum length of fencing required, and show that this is a minimum.

Solution
(i) Area $= xy$
Length of fencing $= x + 2y$

(ii) $50 = xy \Rightarrow y = \frac{50}{x}$

$L = x + 2y$

$= x + \frac{100}{x}$

(ii) $\frac{dL}{dx} = 1 - \frac{100}{x^2}$

At minimum value of L, $\frac{dL}{dx} = 0 \Rightarrow 1 - \frac{100}{x^2} = 0$

$\Rightarrow x^2 = 100$

$\Rightarrow x = 10$

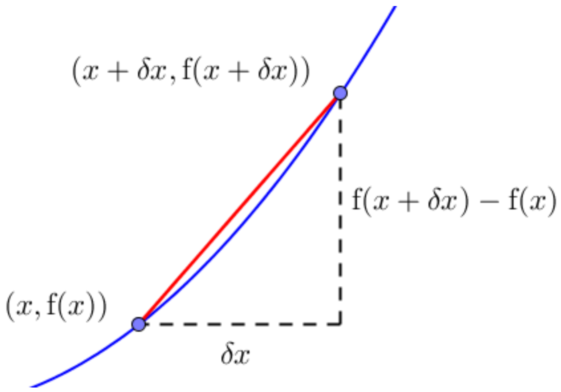
When $x = 10$, $L = 10 + \frac{100}{10} = 20$

$\frac{d^2L}{dx^2} = \frac{200}{x^3}$

When $x = 10$, $\frac{d^2L}{dx^2} > 0$ so this is a minimum.
The minimum length of fencing required is 20m.

Differentiation from first principles

The notation $\frac{dy}{dx}$ for the derivative of a function comes from *differentiation from first principles*. To find the gradient function of a function f, you find the gradient of the chord which joins the point $(x, f(x))$ to another point $(x + \delta x, f(x + \delta x))$, where δx is very small.



When you have simplified this as much as possible, you then let $\delta x = 0$, and the chord becomes a tangent to the graph.

Example 5

Differentiate $y = x^3 - 2x$ from first principles.

Solution

$f(x) = x^3 - 2x$

Need to find the gradient of the chord joining the points $(x, f(x))$ and $(x + \delta x, f(x + \delta x))$

$$\text{Gradient} = \frac{f(x+\delta x) - f(x)}{(x+\delta x) - x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f(x+\delta x) = (x+\delta x)^3 - 2(x+\delta x) = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2x - 2\delta x$$

$$\text{So } f(x+\delta x) - f(x) = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2x - 2\delta x - (x^3 - 2x) = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2\delta x$$

$$\text{So gradient} = \frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2\delta x}{\delta x} = 3x^2 + 3x\delta x + (\delta x)^2 - 2$$

Now letting $\delta x \to 0$, $\frac{dy}{dx} = 3x^2 - 2$

Notice that this is the same answer you would obtain by applying the rules to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

By differentiating from first principles you can see that the rules do indeed give the correct gradient functions (derivatives).

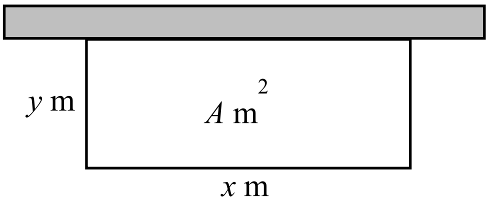
Section 4: More about differentiation

Exercise level 1

1. Find $\frac{d^2y}{dx^2}$ for each of the following.
   (i) $y = x^3 - 3x^2 + 4x - 1$
   (ii) $y = \frac{1}{x} - \frac{2}{x^2}$
   (iii) $y = 2\sqrt{x}$

2. A curve has equation $y = x^3 - 3x^2 + 6$.
   (i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
   (ii) Find the coordinates of any turning points and use $\frac{d^2y}{dx^2}$ to determine the nature of the turning points.
   (iii) Sketch the curve.

3. A farmer has 100 m of fence available, with which he intends to build a pen for his sheep. He intends to create a rectangular pen against a permanent stone wall, as in the diagram.



   (i) Show that $A = \frac{1}{2}x(100 - x)$.
   (ii) Find $\frac{dA}{dx}$ and $\frac{d^2A}{dx^2}$.
   (iii) Find the value of x that makes the area as large as possible, and explain how you know that this is a maximum.

4. Find the gradient of the chord joining the point with x-coordinate 1 to the point with x-coordinate 1 + h on the curve $y = x^2 - 3x + 1$.

5. The point P on the curve $y = 2x^2 - x - 1$ has x-coordinate 1.
   (i) Find the gradient of the chord joining P to the point on the curve with x-coordinate 1 + h.
   (ii) Hence find the gradient of the tangent to the curve at P.

Section 4: More about differentiation

Solutions to Exercise level 1

1. (i) $y = x^3 - 3x^2 + 4x - 1$

$\frac{dy}{dx} = 3x^2 - 6x + 4$

$\frac{d^2y}{dx^2} = 6x - 6$

(ii) $y = \frac{1}{x} - \frac{2}{x^2} = x^{-1} - 2x^{-2}$

$\frac{dy}{dx} = -x^{-2} + 4x^{-3}$

$\frac{d^2y}{dx^2} = -2x^{-3} - 12x^{-4}$

(iii) $y = 2\sqrt{x} = 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = x^{-\frac{1}{2}}$

$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$

2. (i) $y = x^3 - 3x^2 + 6$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{d^2y}{dx^2} = 6x - 6$

(ii) At turning points, $\frac{dy}{dx} = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

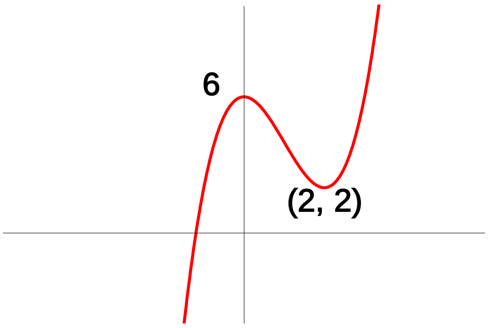
$x = 0 \text{ or } x = 2$

When x = 0, y = 6
When x = 2, $y = 2^3 - 3\times2^2 + 6 = 8 - 12 + 6 = 2$
The turning points are (0, 6) and (2, 2).

When x = 0, $\frac{d^2y}{dx^2} = 0 - 6 < 0$, so (0, 6)is a maximum point.

When x = 2, $\frac{d^2y}{dx^2} = 12 - 6 > 0$, so (2, 2) is a minimum point.

(iii)



3. (i) $x + 2y = 100 \Rightarrow y = \frac{1}{2}(100 - x)$
$A = xy$
$= \frac{1}{2}x(100 - x)$

(ii) $A = 50x - \frac{1}{2}x^2$
$\Rightarrow \frac{dA}{dx} = 50 - x$
$\Rightarrow \frac{d^2A}{dx^2} = -1$

(iii)At maximum, $\frac{dA}{dx} = 0 \Rightarrow x = 50$

and $\frac{d^2A}{dx^2} < 0$, so $x = 50$ gives a maximum

and the area $= 50 \times 25 = 1250$

4. $y = x^2 - 3x + 1$
When x = 1, y = -1
When x = 1 + h, $y = (1 + h)^2 - 3(1 + h) + 1$
$= 1 + 2h + h^2 - 3 - 3h + 1$
$= h^2 - h - 1$

Gradient of chord $= \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{h^2 - h - 1 - (-1)}{1 + h - 1}$
$= \frac{h^2 - h}{h}$
$= h - 1$

5. (i) $y = 2x^2 - x - 1$

When $x = 1$, $y = 0$

When $x = 1 + h$, $y = 2(1+h)^2 - (1+h) - 1$

$$= 2 + 4h + 2h^2 - 1 - h - 1$$

$$= 2h^2 + 3h$$

Gradient of chord $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{2h^2 + 3h - 0}{1 + h - 1}$$

$$= \frac{2h^2 + 3h}{h}$$

$$= 2h + 3$$

(ii) As $h \to 0$, gradient of chord $\to 3$.

So the gradient of the tangent at P is 3.

Section 4: More about differentiation

Exercise level 2

1. A variable rectangle has a constant perimeter of 20 cm. Find the lengths of the sides when the area is a maximum.

2. A square of side x cm is cut from the corners of a piece of card 15 cm by 24 cm. The card is then folded to form an open box.
   (i) Show that the volume of the box is $4x^3 - 78x^2 + 360x$ cm^3.
   (ii) Find a value for x that will make the volume a maximum.

3. A cylinder is cut from a solid sphere of radius 3cm. The height of the cylinder is $2h$.
   (i) Find the radius of the cylinder in terms of h.
   (ii) Show that the volume of the cylinder is $V = 2\pi h(9 - h^2)$.
   (iii) Find the maximum volume of the cylinder as h varies.

4. A cylindrical oil storage tank of radius r and height h is made so that the sum of its radius and its height is 24 m.
   Find the maximum volume of the storage tank.

5. A cylindrical can with height h metres and radius r metres has a capacity of 2 litres.
   (i) Find an expression for h in terms of r.
   (ii) Hence find an expression for the surface area of the can in terms of r only.
   (iii)Find the value of r which minimises the surface area of the can.

6. Find the gradient of the chord joining the point with x-coordinate -2 to the point with x-coordinate -2 + h on the curve $y = x^3 + 2x^2$.

7. The point P on the curve $y = 1 - x - x^3$ has x-coordinate -1.
   (i) Find the gradient of the chord joining P to the point on the curve with x-coordinate -1 + h.
   (ii) Hence find the gradient of the tangent to the curve at P.

8. Differentiate from first principles to find the derivative of each of the functions below.
   (i) $f(x) = 2x^2 - 3x + 1$
   (ii) $f(x) = x^3 - 2x^2 + 3$

Section 4: More about differentiation

Solutions to Exercise level 2

1. Let the length of the sides be x and y.

Considering the perimeter: $2(x+y)=20 \quad \Rightarrow x+y=10$

Let the area be A: $A = xy$

$$= x(10-x)$$

$$= 10x - x^2$$

$$\frac{dA}{dx} = 10 - 2x$$

At turning point, $10-2x=0$

$$2x = 10$$

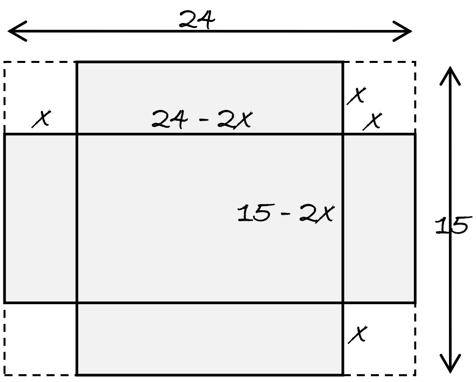
$$x = 5$$

When $x = 5$, $y = 10 - 5 = 5$.

$\frac{d^2A}{dx^2} = -2$ so turning point is a maximum.

The area is a maximum when the lengths of the sides are 5 cm (i.e. the rectangle is a square).

2. (i)



Height of box is x cm

Length of box is (24 – 2x) cm

Width of box is (15 – 2x) cm

Volume $V = x(15-2x)(24-2x)$

$$= x(360 - 78x + 4x^2)$$

$$= 4x^3 - 78x^2 + 360x$$

(ii) $\frac{dV}{dx} = 12x^2 - 156x + 360$

At turning points, $12x^2 - 156x + 360 = 0$

$$x^2 - 13x + 30 = 0$$
$$(x-3)(x-10) = 0$$
$$x = 3 \text{ or } x = 10$$

$x = 10$ is not possible since this would mean that the width would be negative.

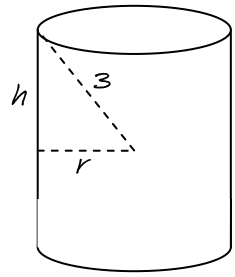
$$\frac{d^2V}{dx^2} = 24x - 156$$

When $x = 3$, $\frac{d^2V}{dx^2} = 72 - 156 < 0$, so $x = 1$ is a maximum point.

The volume of the box is maximised when $x = 3$.

(iii) Volume when $x = 3$ is $V = 3 \times 9 \times 18 = 486$ cm³

3. (i)



$$r^2 + h^2 = 3^2$$
$$r = \sqrt{9 - h^2}$$

(ii) Volume $V = \pi r^2 h$

$$= \pi(9 - h^2) \times 2h$$
$$= 2\pi h(9 - h^2)$$

(iii) $V = 18\pi h - 2\pi h^3$

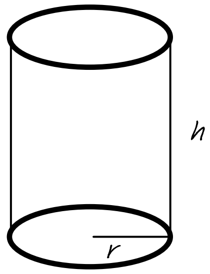
$$\frac{dV}{dh} = 18\pi - 6\pi h^2$$

At turning points, $18\pi - 6\pi h^2 = 0$

$$3 - h^2 = 0$$
$$h = \sqrt{3}$$

$\frac{d^2V}{dh^2} = -12\pi h$ so $h = \sqrt{3}$ is a maximum point.

Maximum volume $V = 2\pi\sqrt{3}(9-3) = 12\pi\sqrt{3}$ cm³.



4. $r + h = 24 \Rightarrow h = 24 - r$

$V = \pi r^2 h = \pi r^2 (24 - r)$

$V = 24\pi r^2 - \pi r^3$

$\Rightarrow \dfrac{dV}{dr} = 48\pi r - 3\pi r^2$

$\Rightarrow \dfrac{d^2V}{dr^2} = 48\pi - 6\pi r$

$\dfrac{dV}{dr} = 0 \;\Rightarrow 3\pi r(16 - r) = 0$

$\Rightarrow r = 0$ (discard!) or $r = 16$

When $r = 16$, $\dfrac{d^2V}{dr^2} < 0$

so $r = 16$ gives the maximum volume, when $h = 8$

and volume $= 2048\pi$ m^3.

5. (i) Volume of a cylindrical can $= \pi r^2 \times h$

2 litres $= 0.002 m^3$ $(1 m^3 = 1000 litres)$

$0.002 = \pi r^2 h$

$h = \dfrac{0.002}{\pi r^2}$

(ii) Surface area $= 2\pi rh + 2\pi r^2$

$= 2\pi r(h + r)$

$= 2\pi r(\dfrac{0.002}{\pi r^2} + r)$

$= 2\pi(\dfrac{0.002}{\pi r} + r^2)$

(iii) $s = 2\pi(\dfrac{0.002}{\pi r} + r^2)$

$= \dfrac{0.004}{r} + 2\pi r^2 = 0.004r^{-1} + 2\pi r^2$

$\dfrac{ds}{dr} = -0.004r^{-2} + 4\pi r$

$\dfrac{ds}{dr} = 0 \;\Rightarrow -0.004r^{-2} + 4\pi r = 0$

$\Rightarrow 4\pi r = 0.004r^{-2}$

$\Rightarrow 4\pi r^3 = 0.004$

$\Rightarrow r^3 = \dfrac{0.004}{4\pi}$

$\Rightarrow r = 0.0683$ m (3 s.f.)

$\frac{d^2s}{dr^2} = 0.008r^{-3} + 4\pi$, which is positive for all values of r, so the stationary point must be a minimum point.

6. $y = x^3 + 2x^2$

When $x = -2$, $y = 0$

When $x = -2 + h$, $y = (h-2)^3 + 2(h-2)^2$

$= h^3 - 6h^2 + 12h - 8 + 2h^2 - 8h + 8$

$= h^3 - 4h^2 + 4h$

Gradient of chord $= \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{h^3 - 4h^2 + 4h - 0}{-2 + h - (-2)}$

$= \frac{h^3 - 4h^2 + 4h}{h}$

$= h^2 - 4h + 4$

7. (i) $y = 1 - x - x^3$

When $x = -1$, $y = 3$

When $x = -1 + h$, $y = 1 - (h-1) - (h-1)^3$

$= 1 - h + 1 - h^3 + 3h^2 - 3h + 1$

$= 3 - 4h + 3h^2 - h^3$

Gradient of chord $= \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{3 - 4h + 3h^2 - h^3 - 3}{-1 + h - (-1)}$

$= \frac{-h^3 + 3h^2 - 4h}{h}$

$= -h^2 + 3h - 4$

(ii) As $h \to 0$, gradient of chord $\to$ -4.

So the gradient of the tangent at P is -4.

8. (i) $f(x) = 2x^2 - 3x + 1$

$f(x+h) = 2(x+h)^2 - 3(x+h) + 1$

$= 2x^2 + 4xh + 2h^2 - 3x - 3h + 1$

$$\text{Gradient of chord } = \frac{f(x+h)-f(x)}{(x+h)-x}\infty$$
$$= \frac{2x^2+4xh+2h^2-3x-3h+1-(2x^2-3x+1)}{x+h-x}$$
$$= \frac{4xh+2h^2-3h}{h}$$
$$= 4x+2h-3$$

As $h \to 0$, gradient of chord $\to 4x-3$.
So $f'(x) = 4x-3$.

(ii) $f(x) = x^3 - 2x^2 + 3$

$$f(x+h) = (x+h)^3 - 2(x+h)^2 + 3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 + 3$$
$$\text{Gradient of chord } = \frac{f(x+h)-f(x)}{(x+h)-x}$$
$$= \frac{x^3+3x^2h+3xh^2+h^3-2x^2-4xh-2h^2+3-(x^3-2x^2+3)}{x+h-x}$$
$$= \frac{3x^2h+3xh^2+h^3-4xh-2h^2}{h}$$
$$= 3x^2+3xh+h^2-4x-2h$$

As $h \to 0$, gradient of chord $\to 3x^2-4x$.
So $f'(x) = 3x^2-4x$.

AQA AS Mathematics Differentiation

Topic Assessment

1. Differentiate with respect to x:

   (i) $y = x^4 + 2x - 3x^3$ (ii) $y = \dfrac{3+2x}{\sqrt{x}}$ (iii) $y = \dfrac{(x-2)(2x+1)}{x^4}$

   [8]

2. Given that $x = (3u+2)(u^2-3)$, find $\dfrac{d^2x}{du^2}$ in terms of u. [3]

3. A curve has equation $y = 2x^3 - 3x^2 - 8x + 9$.
   (i) Find the equation of the tangent to the curve at the point P (2, -3). [3]
   (ii) Find the coordinates of the point Q at which the tangent is parallel to the tangent at P. [3]

4. (i) Find the equation of the normal to the curve $y = x - \dfrac{2}{x^2}$ at the point where $x = 1$. [3]
   (ii) Show that the normal does not meet the curve again. [5]

5. A curve has equation $y = 2x^3 - 6x$.
   (i) Find the points where the curve crosses the x-axis. [2]
   (ii) Find $\dfrac{dy}{dx}$. Hence find the stationary points on the curve. [3]
   (iii) Find $\dfrac{d^2y}{dx^2}$, and use this to determine the nature of the stationary points. [3]
   (iv) Sketch the curve. [2]

6. Find the stationary points of the graph $y = \dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{1}{x^3}$ and determine the nature of each. [7]

7. Two real numbers x and y are such that $2x + y = 100$. Find the maximum value of the product of the two numbers. [5]

8. A cuboid has a square base of length x cm and height y cm.
   (i) Show that the surface area of the cuboid is given by $2x^2 + 4xy$. [2]
   The surface area of the cuboid is 24 cm^2.
   (ii) Show that the volume V of the cuboid is given by $V = 6x - 0.5x^3$. [3]
   (iii) Show that the maximum volume of the box occurs when $x = 2$, and find this maximum volume. [4]

9. Use differentiation from first principles to show that the derivative of x^4 is $4x^3$.
   [4]

Total 60 marks

Summary sheet: Differentiation

G1 Understand and use the derivative of f(x) as the gradient of the tangent to the graph of y = f(x) at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second derivatives; differentiation from first principles for small positive integer powers of x, Understand and use the second derivative as the rate of change of gradient
G2 Differentiate x^n, for rational values of n, and related constant multiples, sums and differences
G3 Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points, Identify where functions are increasing or decreasing

Notation:

There are a few ways of denoting differentiation. You might see all of the following:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \qquad y' \qquad \mathrm{f}'(x)$$

These are all ways of showing that something has been differentiated.

Gradients

- Differentiation is all about gradients. When you differentiate a function you have found the gradient function $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ which **tells you the gradient at any point.**

- Remember that the gradient of the curve is the same as the gradient of the tangent at that point.

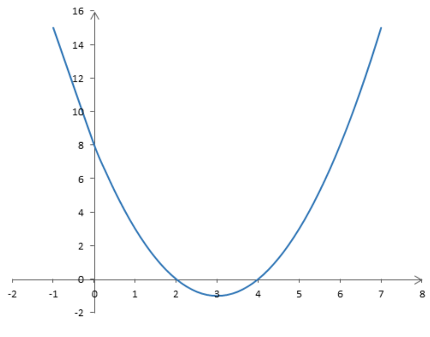
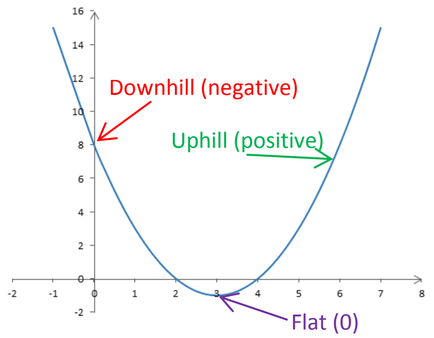
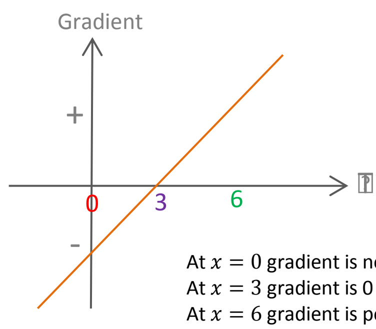
- If $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ the function in increasing (uphill) and if $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ the function in decreasing (downhill).

- The gradient function $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ measures the rate of change of y with respect to x $\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{x\to 0}\frac{\delta y}{\delta x}\right)$

- The gradient of a tangent at the point A on a curve = the limit of the gradient of the chord AP as P moves toward A along the curve. *(Imagine point A on a curve, you draw another point (P) and find the gradient of the line that joins them. You could move P closer and closer to A and the gradient would keep changing slightly, but the closer you get to A the more accurate the gradient is for point A).*

- To sketch the gradient function for a given curve, think about what is happening to the gradient at various points and sketch them.

e.g. Sketch the gradient of the given curve:

Curve	Think about the gradients	Gradient function
		 At $x = 0$ gradient is negative At $x = 3$ gradient is 0 At $x = 6$ gradient is positive

- If you differentiate a second time you have found the rate of change of the gradient – this is called the **second derivative** and can be used to find the type of turning point that you have. The second derivative is denoted $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or y'' or $\mathrm{f}''(x)$

Summary sheet: Differentiation

Differentiating from first principles

Remember that differentiation involves small changes in x (δx) and small changes in y (δy). So to differentiate from first principles just replace every x with $(x + \delta x)$ and every y with $(y + \delta y)$.

e.g. Differentiate $y = x^2 + 4x$ from first principles.

$$y = x^2 + 4x$$

Add the small changes
$$y + \delta y = (x + \delta x)^2 + 4(x + \delta x) \qquad (1)$$

Original
$$y = x^2 + 4x \qquad (2)$$

Subtract (1-2) to leave δy on its own
$$\delta y = (x + \delta x)^2 + 4(x + \delta x) - x^2 - 4x$$

Divide by δx to get $\frac{\delta y}{\delta x}$
$$\frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 + 4(x + \delta x) - x^2 - 4x}{\delta x}$$

Now tidy up & simplify
$$= \frac{x^2 + 2x\delta x + (\delta x)^2 + 4x + 4\delta x - x^2 - 4x}{\delta x}$$
$$= \frac{2x\delta x + (\delta x)^2 + 4\delta x}{\delta x}$$
$$= 2x + \delta x + 4$$

So as $\delta x \to 0$ we have
$$\frac{\delta y}{\delta x} \to \frac{dy}{dx} = 2x + 4$$

Differentiating

The good news is that there is a much easier way to differentiate without using first principles.

Original Function: $\boldsymbol{y = kx^n}$

Gradient function: $\boldsymbol{\frac{dy}{dx} = knx^{n-1}}$

Multiply the expression by the power then reduce the power by 1.

For functions with more than one term, differentiate each term separately. Remember for each term: **expression x power**, then **power -1**.

e.g. **Differentiate $y = 5x^3 - 3x^2 + 7x - 5$**

$$\frac{dy}{dx} = 15x^2 - 6x + 7$$

Summary sheet: Differentiation

Finding a gradient

Once you have found the gradient function you can use it to find the gradient at any point.

e.g. **Find the gradient of the curve $\boldsymbol{y = 5x^3 - 3x^2 + 7x - 5}$ at the point $\boldsymbol{x = 3}$**

We already know that the gradient function is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^2 - 6x + 7$$

So the gradient when $x = 3$ can be found by:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 15(3)^2 - 6(3) + 7 \\ &= \mathbf{124}\end{aligned}$$

Tangent and Normal

Remember that a **tangent** to a curve will have the **same** gradient and the **normal** will have a **perpendicular** gradient. So to find either of them you will need to start off with finding the gradient of the curve at the point you are interested in. Remember that both the tangent and the normal are straight lines and so they will be of the form $y = mx + c$ and you will need to find m (gradient) and c.

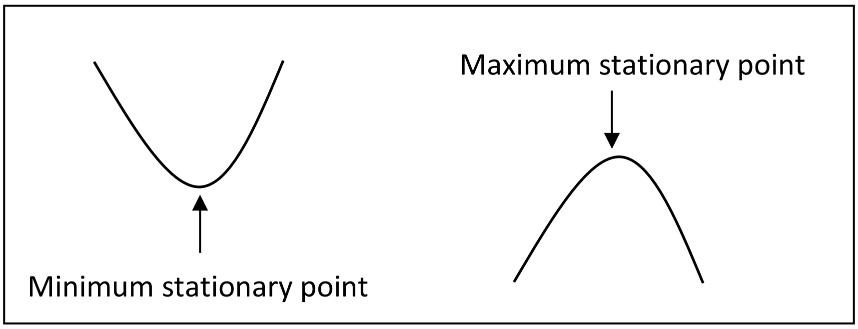
e.g. Find the tangent and the normal to the curve $\boldsymbol{y = 3x^2 - 4x + 2}$ at the point $\boldsymbol{x = 2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x - 4$$

So when $x = 2$ the gradient $= 6(2) - 4 = \mathbf{8}$
We also know that when $x = 2,\ y = 3(2)^2 - 4(2) + 2 = 6$

	Gradient	Equation	Substitute point $(2, 6)$ to find c	Equation
Tangent (same gradient)	8	$y = 8x + c$	$c = -10$	$y = 8x - 10$
Normal (perpendicular gradient)	$-\frac{1}{8}$	$y = -\frac{1}{8}x + c$	$c = \frac{25}{4}$	$y = -\frac{1}{8}x + \frac{25}{4}$

Stationary Points (maximum or minimum)



Summary sheet: Differentiation

Looking at the sketches you can see that at both the maximum and the minimum the gradient is 0. So to find where the stationary point is you need to find where the gradient=0. To decide what type of stationary point it is (max or min) you would use the second derivative (i.e. differentiate again).

1st derivative: tells you **where** the stationary point(s) is/are.

Put $\frac{dy}{dx} = 0$ and solve to find the value(s) of x.

Then substitute x into the original equation to find the y coordinate(s).

2nd derivative: tells you the **type** of stationary point(s) you have found.
Substitute your value(s) of x into the 2nd derivative and:

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ it's a minimum (**positive** is **minimum**).

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ it's a maximum (**negative** is **maximum**).

e.g. for the curve $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 11$ find the stationary point and decide whether it is a maximum or minimum.

$$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 11$$

Find **where** the stationary point(s) is(are):

Differentiate once: $\frac{dy}{dx} = x^2 + x - 6$

Put $\frac{dy}{dx} = 0$: $x^2 + x - 6 = 0$

Solve (factorise or use quadratic formula) to get $\boldsymbol{x = 2}$ or $\boldsymbol{x = -3}$

Find **type** of stationary points:

Differentiate again: $\frac{d^2y}{dx^2} = 2x + 1$

When $x = 2$: $\frac{d^2y}{dx^2} = 2(2) + 1 = 5$ Positive so ***Minimum***

When $x = -3$: $\frac{d^2y}{dx^2} = 2(-3) + 1 = -5$ Negative so ***Maximum***

Section 1: Introduction to integration

These notes contain subsections on:

- Reversing differentiation
- The rule for integrating x^n
- Indefinite integrals: formal notation
- Finding the arbitrary constant

Reversing differentiation

Integration is the reverse of differentiation. If you are given an expression for $\frac{dy}{dx}$, and you want to find an expression for y, you need to use integration. This is sometimes called solving a differential equation.

Remember that when you integrate, you must always add an arbitrary constant (see the textbook for the explanation of this).

Example 1 shows how you can integrate a function by thinking about what function you would need to differentiate to obtain the given function.

Example 1

Find y as a function of x for each of the following.

(i) $\frac{dy}{dx} = x^3$ (ii) $\frac{dy}{dx} = x^6$ (iii) $\frac{dy}{dx} = x$

(iv) $\frac{dy}{dx} = 2x^2$ (v) $\frac{dy}{dx} = 3x^4$ (vi) $\frac{dy}{dx} = 5$

Solution

(i) $\frac{dy}{dx} = x^3 \Rightarrow y = \frac{1}{4}x^4 + c$

The derivative of x^4 is $4x^3$.
So integrating $4x^3$ gives x^4.
Therefore integrating x^3 gives $\frac{1}{4}x^4$.

(ii) $\frac{dy}{dx} = x^6 \Rightarrow y = \frac{1}{7}x^7 + c$

The derivative of x^7 is $7x^6$.
So integrating $7x^6$ gives x^7.
Therefore integrating x^6 gives $\frac{1}{7}x^7$.

(iii) $\frac{dy}{dx} = x \Rightarrow y = \frac{1}{2}x^2 + c$

The derivative of x^2 is $2x$.
So integrating $2x$ gives x^2.
Therefore integrating x gives $\frac{1}{2}x^2$.

(iv) $\frac{dy}{dx} = 2x^2 \Rightarrow y = \frac{2}{3}x^3 + c$

The derivative of x^3 is $3x^2$.
So integrating $3x^2$ gives x^3.
Therefore integrating x^2 gives $\frac{1}{3}x^3$,
and integrating $2x^2$ gives $\frac{2}{3}x^3$.

(v) $\frac{dy}{dx} = 3x^4 \Rightarrow y = \frac{3}{5}x^5 + c$

The derivative of x^5 is $5x^4$.
So integrating $5x^4$ gives x^5.
Therefore integrating x^4 gives $\frac{1}{5}x^5$,
and integrating $3x^4$ gives $\frac{3}{5}x^5$.

(vi) $\frac{dy}{dx} = 5 \Rightarrow y = 5x + c$

The derivative of x is 1.
So integrating 1 gives x.
Therefore integrating 5 gives $5x$.

The rule for integrating x^n

The method for integrating any polynomial function can be summed up as:

- Integrating x^n, where n is a positive integer, gives $\frac{x^{n+1}}{n+1}$
- Integrating kx^n, where n is a positive integer and k is a constant, gives $\frac{kx^{n+1}}{n+1}$
- You can integrate the sum of any number of such functions by simply integrating one term at a time.

Indefinite integration: formal notation

In Example 1 an expression for $\frac{dy}{dx}$ was given and used to find an expression for y.
So you would write:

$$\frac{dy}{dx} = 2x \Rightarrow y = x^2 + c.$$

Using the formal notation, you would write this as:

$$\int 2x \, dx = x^2 + c$$

You would read this as "the integral of $2x$ with respect to x"

The next example shows integration expressed using formal notation.

Example 2

Integrate each of the following functions.

(i) $x^3 + 3x + 2$

(ii) $4x^2 - 5x - 1$

(iii) $(x+3)(x-2)$

Remember the arbitrary constant

Solution

(i) $\int \left(x^3 + 3x + 2\right) dx = \frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x + c$

(ii) $\int \left(4x^2 - 5x - 1\right) dx = \frac{4}{3}x^3 - \frac{5}{2}x^2 - x + c$

(iii) $\int (x+3)(x-2) \, dx = \int \left(x^2 + x - 6\right) dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + c$

Finding the arbitrary constant

If you are given additional information, you can find the value of the arbitrary constant by substituting the given information. This is sometimes called finding the particular solution of a differential equation. The next example shows how this is done.

Example 3

The gradient of a curve at any point (x, y) is given by $\frac{dy}{dx} = x^2(2x + 1)$.

The curve passes through the point (1, 5).
Find the equation of the curve.

Solution:

$\frac{dy}{dx} = x^2(2x + 1) = 2x^3 + x^2$

Just as with differentiating, you need to expand the brackets first

Integrating: $y = 2 \times \frac{1}{4}x^4 + \frac{1}{3}x^3 + c$

$= \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

When $x = 1$, $y = 5$ $\Rightarrow 5 = \frac{1}{2} + \frac{1}{3} + c$

$\Rightarrow c = 5 - \frac{1}{2} - \frac{1}{3} = \frac{25}{6}$

Substitute the given values of x and y

So the equation of the curve is $y = \frac{1}{2}x^4 + \frac{1}{3}x^3 + \frac{25}{6}$

Section 1: Introduction to integration

Exercise level 1

1. Find the following indefinite integrals.
   (i) $\int (2x+3)\,\mathrm{d}x$
   (ii) $\int (x^2-4x-1)\,\mathrm{d}x$
   (iii) $\int (x^5+1)\,\mathrm{d}x$
   (iv) $\int (x^3+2x-7)\,\mathrm{d}x$

2. A curve has gradient function $\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2-4$.
   (i) Find an expression for y in terms of x.
   (ii) Find the particular curve that passes through the point (2, -1).
   (iii) Show that this curve also passes through the point (1, -4).

3. The gradient function of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x}=4x-x^2$. Find the equation of the curve given that it passes through the point (3, 2).

4. A stone is thrown vertically upwards such that $\frac{\mathrm{d}h}{\mathrm{d}t}=25-10t$, where t is the time in seconds and h is the height of the stone in metres.
   Given that when $t = 0$, $h = 30$, find the value of t for which $h = 0$.

5. Find y in terms of x given that $\frac{\mathrm{d}y}{\mathrm{d}x}=(x+1)^2$ and that $y = 0$ when $x = 2$.

Section 1: Introduction to integration

Solutions to Exercise level 1

1. (i) $\int(2x+3)dx = x^2+3x+c$

(ii) $\int\left(x^2-4x-1\right)dx = \frac{1}{3}x^3-2x^2-x+c$

(iii) $\int\left(x^5+1\right)dx = \frac{1}{6}x^6+x+c$

(iv) $\int\left(x^3+2x-7\right)dx = \frac{1}{4}x^4+x^2-7x+c$

2. (i) $\frac{dy}{dx} = 3x^2-4$

$y = 3\times\frac{1}{3}x^3-4x+c$

$= x^3-4x+c$

(ii) When $x = 2$, $y = -1$

$-1 = 2^3-4\times2+c$

$-1 = 8-8+c$

$c = -1$

Equation of curve is $y = x^3-4x-1$

(iii) When $x = 1$, $y = 1^3-4\times1-1 = 1-4-1 = -4$

so the curve passes through the point (1, -4).

3. $\frac{dy}{dx} = 4x-x^2$

$y = 4\times\frac{1}{2}x^2-\frac{1}{3}x^3+c$

$= 2x^2-\frac{1}{3}x^3+c$

When $x = 3$, $y = 2$

$2 = 2\times3^2-\frac{1}{3}\times3^3+c$

$2 = 18-9+c$

$c = 2-18+9 = -7$

$y = 2x^2-\frac{1}{3}x^3-7$

4. $\frac{dh}{dt} = 25 - 10t$

$h = 25t - 5t^2 + c$

When t = 0, h = 30 $\Rightarrow 30 = c$

$h = 25t - 5t^2 + 30$

When h = 0, $25t - 5t^2 + 30 = 0$

$5t - t^2 + 6 = 0$

$t^2 - 5t - 6 = 0$

$(t - 6)(t + 1) = 0$

$t = 6$ or $t = -1$

Since t must be positive, the value of t must be 6.

5. $\frac{dy}{dx} = (x+1)^2 = x^2 + 2x + 1$

$y = \frac{1}{3}x^3 + x^2 + x + c$

When x = 2, y = 0

$0 = \frac{1}{3} \times 2^3 + 2^2 + 2 + c$

$0 = \frac{8}{3} + 4 + 2 + c$

$c = -\frac{26}{3}$

$y = \frac{1}{3}x^3 + x^2 + x - \frac{26}{3}$

Section 1: Introduction to integration

Exercise level 2

1. The gradient function of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^2 + x$.

   (i) Find the equation of the curve given that $y = 2$ when $x = 1$.

   (ii) Find the value of y when $x = 3$.

2. The gradient of a curve at the point (x, y) is given by $4(1 - x)$. Given that the curve has a maximum value of 8, find the equation of the curve.

3. Find an expression for y in terms of x if $\frac{\mathrm{d}y}{\mathrm{d}x} = (x-1)(3x-5)$ and $y = 2$ when $x = 1$

   .

4. A curve with gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^2 - 1$ has a local minimum value of 1. Find the equation of the curve and the coordinates of the local maximum value.

5. A curve has gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x + k$

   (i) It has a maximum point at $x = -2$. Find the value of k.

   (ii) The curve passes through the point (1, 3). Find the equation of the curve.

Section 1: Introduction to integration

Solutions to Exercise level 2

1. (i) $\frac{dy}{dx} = 4x^2 + x$

$y = \frac{4}{3}x^3 + \frac{1}{2}x^2 + c$

When $x = 1$, $y = 2$

$2 = \frac{4}{3}\times 1^3 + \frac{1}{2}\times 1^2 + c$

$c = 2 - \frac{4}{3} - \frac{1}{2} = \frac{1}{6}$

$y = \frac{4}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}$

(ii) When $x = 3$, $y = \frac{4}{3}\times 3^3 + \frac{1}{2}\times 3^2 + \frac{1}{6}$

$= 36 + \frac{9}{2} + \frac{1}{6}$

$= 40\frac{2}{3}$

2. $\frac{dy}{dx} = 4(1-x)$

At maximum point, $\frac{dy}{dx} = 0 \quad \Rightarrow x = 1$

So the curve passes through the point (1, 8).

$\frac{dy}{dx} = 4(1-x) = 4 - 4x$

$y = 4x - 2x^2 + c$

When $x = 1$, $y = 8$

$8 = 4 - 2 + c \Rightarrow c = 6$

The equation of the curve is $y = 4x - 2x^2 + 6$.

3. $\frac{dy}{dx} = (x-1)(3x-5) = 3x^2 - 8x + 5$

$y = 3\times\frac{1}{3}x^3 - 8\times\frac{1}{2}x^2 + 5x + c$

$= x^3 - 4x^2 + 5x + c$

When $x = 1$, $y = 2$

$2 = 1^3 - 4\times 1^2 + 5\times 1 + c$

$c = 2 - 1 + 4 - 5 = 0$

$y = x^3 - 4x^2 + 5x$

4. At turning points, $4x^2 - 1 = 0$

$$(2x-1)(2x+1) = 0$$

$$x = \tfrac{1}{2} \text{ or } -\tfrac{1}{2}$$

For $x < -\tfrac{1}{2}$, $\dfrac{dy}{dx} > 0$

For $-\tfrac{1}{2} < x < \tfrac{1}{2}$, $\dfrac{dy}{dx} < 0$

For $x > \tfrac{1}{2}$, $\dfrac{dy}{dx} > 0$

Therefore there is a maximum point where $x = -\tfrac{1}{2}$ and a minimum point where $x = \tfrac{1}{2}$.

So the graph passes through the point $\left(\tfrac{1}{2}, 1\right)$.

$$\frac{dy}{dx} = 4x^2 - 1$$

$$y = \tfrac{4}{3}x^3 - x + c$$

When $x = \tfrac{1}{2}, y = 1 \Rightarrow 1 = \tfrac{4}{3}(\tfrac{1}{2})^3 - \tfrac{1}{2} + c$

$$\Rightarrow 1 = \tfrac{1}{6} - \tfrac{1}{2} + c$$

$$\Rightarrow c = \tfrac{4}{3}$$

The equation of the curve is $y = \tfrac{4}{3}x^3 - x + \tfrac{4}{3}$.

The maximum point is when $x = -\tfrac{1}{2}$.

$$y = \tfrac{4}{3}(-\tfrac{1}{2})^3 - (-\tfrac{1}{2}) + \tfrac{4}{3}$$

$$= -\tfrac{1}{6} + \tfrac{1}{2} + \tfrac{4}{3}$$

$$= \tfrac{5}{3}$$

5. (i) For $x = -2$, $\dfrac{dy}{dx} = 0 \Rightarrow 12 + 4 + k = 0$

$$\Rightarrow k = -16$$

and so $\dfrac{dy}{dx} = 3x^2 - 2x - 16$

(ii) $\dfrac{dy}{dx} = 3x^2 - 2x + 8 \Rightarrow y = x^3 - x^2 - 16x + c$

and since the curve passes through (1, 3)

$3 = 1 - 1 - 16 + c \Rightarrow c = 19$

so the equation of the curve is $y = x^3 - x^2 - 16x + 19$

Section 2: Finding the area under a curve

Notes and Examples

These notes contain subsections on:

- Definite integration
- The definite integral as an area

Definite integration

The definite integral from a to b of a function f(x), which is written as $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$, is found as follows:

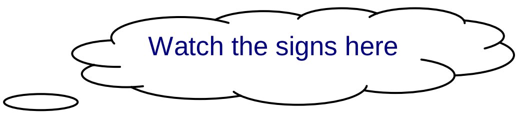
- Integrate f(x) – suppose we call the integral $\mathrm{g}(x)$
- Write the integral in square brackets, with the limits on the right hand side: $[\mathrm{g}(x)]_a^b$
- Work out the value of $\mathrm{g}(x)$ with $x = a$ and $x = b$, and subtract: $[\mathrm{g}(x)]_a^b = \mathrm{g}(b) - \mathrm{g}(a)$.

Example 1

Evaluate $\int_1^2 (2x - x^2)\,\mathrm{d}x$.

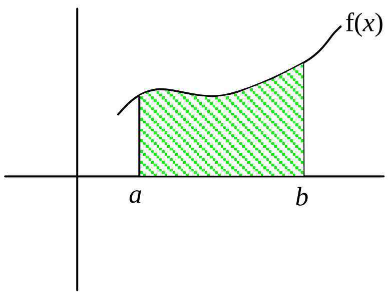
Solution

$$\begin{aligned}\int_1^2 \left(2x - x^2\right)\mathrm{d}x &= \left[x^2 - \tfrac{1}{3}x^3\right]_1^2 \\ &= \left(2^2 - \tfrac{1}{3}\times 2^3\right) - \left(1^2 - \tfrac{1}{3}\times 1^3\right) \\ &= 4 - \tfrac{8}{3} - 1 + \tfrac{1}{3} \\ &= \tfrac{2}{3}\end{aligned}$$



The definite integral as an area

The definite integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ calculates the area between the curve $y = \mathrm{f}(x)$ and the x-axis.



If the curve is above the x-axis, so that the value of y is positive, the definite integral works out to be positive. However, if the curve is below the x-axis, so that y is negative, the integral works out to be negative.

Example 2

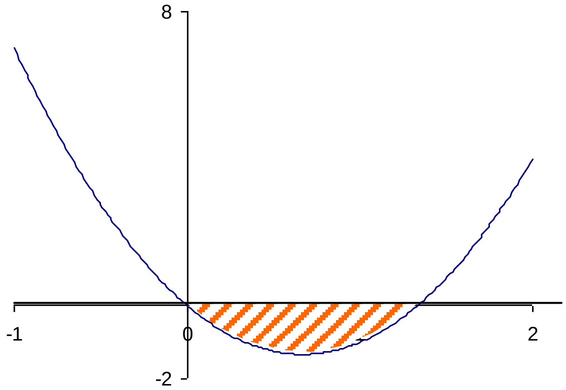
Find the total area enclosed by the curve $y = 3x^2 - 4x$ and the x-axis.

Solution:

The curve crosses the x-axis when $3x^2 - 4x = 0$

$\Rightarrow \quad x(3x-4) = 0$

$\Rightarrow \quad x = 0$ or $x = \frac{4}{3}$



$$\text{Area} = \int_0^{\frac{4}{3}} (3x^2 - 4x)\,dx = \left[x^3 - 2x^2\right]_0^{\frac{4}{3}}$$

$$= \left(\left(\tfrac{4}{3}\right)^3 - 2\left(\tfrac{4}{3}\right)^2\right) - \left(0^3 - 2\times 0^2\right)$$

$$= -\tfrac{32}{27}$$

This integral works out to be negative because the curve is below the x-axis.

The area is $\frac{32}{27}$ square units.

Notice that you should give your final answer as positive, since an area cannot be negative. However, remember that this only applies to definite integrals which are being used to find an area – if you are just asked to work out the value of a definite integral, then the answer may be positive or negative.

Section 2: Area under a curve

Exercise level 1

1. Find the following indefinite integrals.

   (i) $\int 4x^3 \mathrm{d}x$

   (ii) $\int (x^3 - 3x^2) \mathrm{d}x$

   (iii) $\int (10x^4 + 3x^2 + 4) \mathrm{d}x$

   (iv) $\int (3x - 1)^2 \mathrm{d}x$

   (v) $\int x(3x - 4) \mathrm{d}x$

2. Evaluate the following definite integrals.

   (i) $\int_{-1}^{1} (4x + 5) \mathrm{d}x$

   (ii) $\int_{-1}^{0} (6x^2 - 2x) \mathrm{d}x$

   (iii) $\int_{2}^{4} (x^2 - x + 3) \mathrm{d}x$

   (iv) $\int_{-1}^{2} (2 + x - x^2) \mathrm{d}x$

   (v) $\int_{-1}^{2} (x^3 - x + 4) \mathrm{d}x$

3. Find the areas enclosed by the x axis and the following curves.

   (i) $y = (1 - x)(x + 2)$

   (ii) $y = 3x^2 - x^3$

Section 2: Finding the area under a curve

Solutions to Exercise level 1

1. (i) $\int 4x^3 dx = x^4 + c$

(ii) $\int (x^3 - 3x^2) dx = \frac{1}{4}x^4 - x^3 + c$

(iii) $\int (10x^4 + 3x^2 + 4) dx = 2x^5 + x^3 + 4x + c$

(iv) $$\int (3x-1)^2 dx = \int \left(9x^2 - 6x + 1\right) dx$$
$$= 3x^3 - 3x^2 + x + c$$

(v) $$\int x(3x-4)\, dx = \int \left(3x^2 - 4x\right) dx$$
$$= x^3 - 2x^2 + c$$

2. (i) $$\int_{-1}^{1} (4x+5) dx = \left[2x^2 + 5x\right]_{-1}^{1}$$
$$= 2 + 5 - (2 - 5)$$
$$= 7 - (-3)$$
$$= 10$$

(ii) $$\int_{-1}^{0} (6x^2 - 2x) dx = \left[2x^3 - x^2\right]_{-1}^{0}$$
$$= 0 - (-2 - 1)$$
$$= 3$$

(iii) $$\int_{2}^{4} (x^2 - x + 3) dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 3x\right]_{2}^{4}$$
$$= \left(\tfrac{64}{3} - 8 + 12\right) - \left(\tfrac{8}{3} - 2 + 6\right)$$
$$= \tfrac{64}{3} + 4 - \tfrac{8}{3} - 4$$
$$= \tfrac{56}{3}$$

(iv) $$\int_{-1}^{2} (2 + x - x^2) dx = \left[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_{-1}^{2}$$
$$= \left(4 + 2 - \tfrac{8}{3}\right) - \left(-2 + \tfrac{1}{2} + \tfrac{1}{3}\right)$$
$$= 6 - \tfrac{8}{3} + 2 - \tfrac{1}{2} - \tfrac{1}{3}$$
$$= 8 - 3 - \tfrac{1}{2}$$
$$= \tfrac{9}{2}$$

(v) $\int_{-1}^{2}(x^3 - x + 4)dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 + 4x\right]_{-1}^{2}$

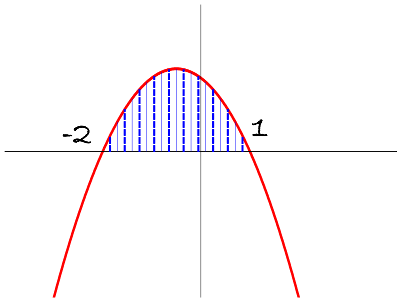
$= (4 - 2 + 8) - (\frac{1}{4} - \frac{1}{2} - 4)$

$= 10 + \frac{1}{4} + 4$

$= 14.25$

3. (i) $y = (1 - x)(x + 2)$

The graph cuts the x-axis at $x = 1$ and $x = -2$.
The coefficient of x^2 is negative, so the graph is "upside down".



Area $= \int_{-2}^{1}(1 - x)(x + 2)dx$

$= \int_{-2}^{1}(2 - x - x^2)dx$

$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1}$

$= (2 - \frac{1}{2} - \frac{1}{3}) - (-4 - 2 + \frac{8}{3})$

$= 2 - \frac{1}{2} - \frac{1}{3} + 6 - \frac{8}{3}$

$= \frac{9}{2}$ square units

(ii) $y = 3x^2 - x^3 = x^2(3 - x)$

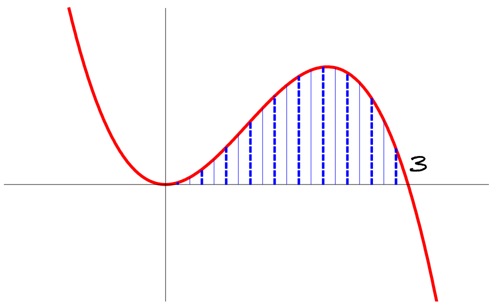
The graph cuts the x-axis at (3, 0) and touches the x-axis at the origin.

Area $= \int_{0}^{3}(3x^2 - x^3)dx$

$= \left[x^3 - \frac{1}{4}x^4\right]_{0}^{3}$

$= 27 - \frac{81}{4} - 0$

$= 6.75$ square units



Section 2: Area under a curve

Exercise level 2

1. Evaluate:
   (i) $\int_{-2}^{2} (x+3)(x-2)\,dx$
   (ii) $\int_{0}^{2} x(x^2+1)\,dx$

2. Find the total area enclosed by $y = x^2 - 2x - 3$, the x axis, $x = -3$ and $x = 3$.

3. Find the total area enclosed by $y = x(x-1)$, the x axis and the line $x = 2$.

4. (i) Sketch the curve $y = x^3 - x$ for values of x from –3 to +3.
   (ii) Find the area bounded by the curve, the x axis and the lines $x = 1$ and $x = 2$.
   (iii) Find the area bounded by the curve and the lines $x = -1$, $x = 0$ and the x axis.
   (iv) From your answers to (ii) and (iii) and your sketch deduce the total area given by the integral $\int_{0}^{2} (x^3 - x)\,dx$. Explain your reasoning.

5. (i) Sketch the general shape of curve $y = x(x+2)(x-3)$ showing where it crosses the x-axis.
   (ii) Find the total area enclosed between the graph and the x-axis.

6. Find the area enclosed by the curve $y = (x-3)^2$ and the lines $y = 0$ and $y = 4$.

7. Find the total area enclosed by the curve $y = x(4 - x^2)$ and the x-axis.

Section 2: Finding the area under a curve

Solutions to Exercise level 2

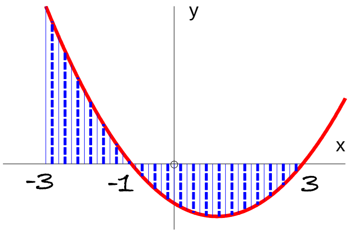
1. (i) $\int_{-2}^{2}(x+3)(x-2)\,dx = \int_{-2}^{2}(x^2+x-6)\,dx$

$$= \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 - 6x\right]_{-2}^{2}$$
$$= \left(\tfrac{8}{3}+2-12\right) - \left(-\tfrac{8}{3}+2+12\right)$$
$$= -\tfrac{56}{3}$$

(ii) $\int_{0}^{2}x(x^2+1)\,dx = \int_{0}^{3}\left(x^3+x\right)dx$

$$= \left[\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right]_{0}^{2}$$
$$= (4+2)-(0+0)$$
$$= 6$$

2. $y = x^2 - 2x - 3 = (x-3)(x+1)$



Area above x-axis $= \int_{-3}^{-1}(x^2-2x-3)\,dx$

$$= \left[\tfrac{1}{3}x^3 - x^2 - 3x\right]_{-3}^{-1}$$
$$= \left(-\tfrac{1}{3}-1+3\right) - \left(-9-9+9\right)$$
$$= \tfrac{32}{3}$$

Area below x-axis $= \left[\tfrac{1}{3}x^3 - x^2 - 3x\right]_{-1}^{3}$

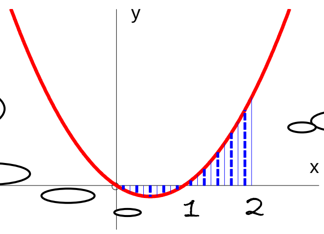
$$= (9-9-9) - \left(-\tfrac{1}{3}-1+3\right)$$
$$= -9 - \tfrac{5}{3}$$
$$= -\tfrac{32}{3}$$

Total area $= \tfrac{32}{3} + \tfrac{32}{3} = \tfrac{64}{3}$ square units

3. $y = x(x-1)$

The graph cuts the x-axis at the origin and the point (1, 0).

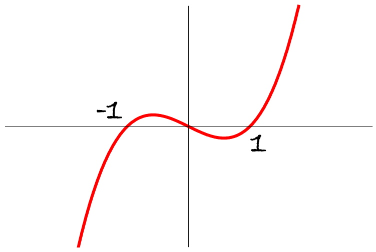
This area is negative as it is below the x-axis.



The areas above and below the x-axis must be calculated separately.

$$\text{Area below x-axis} = \int_0^1 x(x-1)\,dx$$
$$= \int_0^1 (x^2 - x)\,dx$$
$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^1$$
$$= \tfrac{1}{3} - \tfrac{1}{2}$$
$$= -\tfrac{1}{6}$$
$$\text{Area above x-axis} = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_1^2$$
$$= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - \tfrac{1}{2}\right)$$
$$= \tfrac{5}{6}$$
$$\text{Total area} = \tfrac{1}{6} + \tfrac{5}{6} = 1 \text{ square unit.}$$

4. (i) $y = x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$

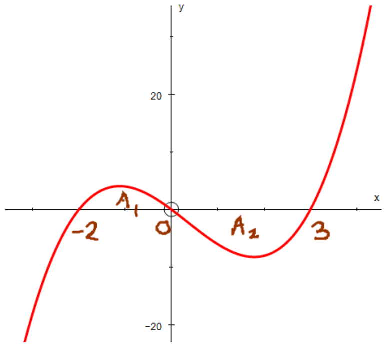


(ii) $$\text{Area} = \int_1^2 (x^3 - x)\,dx$$
$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_1^2$$
$$= (4-2) - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)$$
$$= 2.25 \text{ square units}$$

(iii) $$\text{Area} = \int_{-1}^0 (x^3 - x)\,dx$$
$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_{-1}^0$$
$$= 0 - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)$$
$$= 0.25 \text{ square units}$$

(iv) By symmetry, the area between $x = 0$ and $x = 1$ is the same as the area between $x = -1$ and $x = 0$, only below the x-axis.
So the area between $x = 0$ and $x = 2$ is given by $2.25 + 0.25 = 2.5$ square units.

5. (i)



(ii) $A_1 = \int_{-2}^{0} x(x+2)(x-3)\,dx$

$= \int_{-2}^{0} x^3 - x^2 - 6x\,dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2\right]_{-2}^{0}$

$= 0 - \left(4 + \frac{8}{3} - 12\right)$

$= \frac{16}{3}$

$A_2 = \int_{0}^{3} x(x+2)(x-3)\,dx$

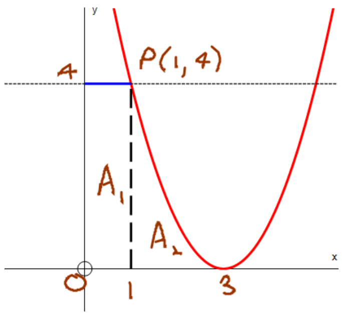
$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2\right]_{0}^{3}$

$= \left(\frac{81}{4} - \frac{27}{3} - 27\right) - (0)$

$= -\frac{63}{4}$

Total area $= \frac{16}{3} - (-\frac{63}{4}) = \frac{253}{12} \approx 21.1$ square units.

6. P is the point given by $4 = (x-3)^2 \Rightarrow x = 1$ or 5
so P $= (1, 4)$



Area = rectangle $+\int_{1}^{3} (x-3)^2\,dx$

$= 4 + \int_{1}^{3} x^2 - 6x + 9\,dx$

$= 4 + \left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_{1}^{3}$

$= 4 + (9 - 27 + 27) - (\frac{1}{3} - 3 + 9)$

$= \frac{20}{3}$ units^2

7. $y = x(4 - x^2) = x(2-x)(2+x)$
When $y = 0$, $x = 0$, $x = -2$, or $x = 2$

$$A_1 = \int_{-2}^{0} 4x - x^3 \, dx$$
$$= \left[2x^2 - \tfrac{1}{4}x^4\right]_{-2}^{0}$$
$$= 0 - (8 - 4) = -4$$
$$A_2 = \int_{0}^{2} 4x - x^3 \, dx$$
$$= \left[2x^2 - \tfrac{1}{4}x^4\right]_{0}^{2}$$
$$= (8 - 4) - 0 = 4$$
$$\text{Area} = 4 + 4 = 8$$

Section 3: Further integration

Notes and Examples

These notes contain subsections on

- Integrating kx^n for negative and fractional n
- Applications of integration

Integrating kx^n for negative and fractional n

In Section 1 you saw that the integral of x^n, where n is a positive integer, is given by

$$\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c \qquad \text{where } c \text{ is an arbitrary constant}$$

In fact this formula is true not only when n is a positive integer, but for all real values of n, including negative numbers and fractions, except for $n = -1$.

The formula does not work for $n = -1$, since this would give a denominator of 0. There is a different way to integrate $\frac{1}{x}$, which is covered in later in A level Mathematics.

Example 1

Find the following indefinite integrals

(i) $\int \sqrt{x}\,\mathrm{d}x$

(ii) $\int \frac{1}{x^3}\,\mathrm{d}x$

(iii) $\int \left(\frac{2}{x^2} - \frac{3}{\sqrt{x}}\right)\mathrm{d}x$

Solution

(i) $\int \sqrt{x}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x$

$= \frac{2}{3}x^{\frac{3}{2}} + c$

$n = \frac{1}{2}$, so $n+1 = \frac{3}{2}$, so divide by $\frac{3}{2}$, i.e. multiply by $\frac{2}{3}$.

(ii) $\int \frac{1}{x^3}\,\mathrm{d}x = \int x^{-3}\,\mathrm{d}x$

$= -\frac{1}{2}x^{-2} + c$

$n = -3$, so $n+1 = -2$, so divide by -2.

(iii) $\displaystyle\int\left(\frac{2}{x^2}-\frac{3}{\sqrt{x}}\right)\mathrm{d}x=\int\left(2x^{-2}-3x^{-\frac{1}{2}}\right)\mathrm{d}x$

$\displaystyle = -2x^{-1}-3\times 2x^{\frac{1}{2}}+c$

$\displaystyle = -\frac{2}{x}-6\sqrt{x}+c$

$n=-2$, so $n+1=-1$, so divide by -1.

$n=-\frac{1}{2}$, so $n+1=\frac{1}{2}$, so divide by $\frac{1}{2}$, i.e. multiply by 2.

Example 2

Find the following definite integrals.

(i) $\displaystyle\int_1^2\left(\frac{4x-1}{x^4}\right)\mathrm{d}x$

(ii) $\displaystyle\int_1^4(3-x)\sqrt{x}\;\mathrm{d}x$

Solution

(i) $\displaystyle\int_1^2\left(\frac{4x-1}{x^4}\right)\mathrm{d}x=\int_1^2\left(4x^{-3}-x^{-4}\right)\mathrm{d}x$

$\displaystyle = \left[4\times -\tfrac{1}{2}x^{-2}+\tfrac{1}{3}x^{-3}\right]_1^2$

$\displaystyle = \left[-\frac{2}{x^2}+\frac{1}{3x^3}\right]_1^2$

$\displaystyle = \left(-\tfrac{1}{2}+\tfrac{1}{24}\right)-\left(-2+\tfrac{1}{3}\right)$

$\displaystyle = \tfrac{29}{24}$

Substitute $x = 2$ in first bracket and $x = 1$ in second bracket

(ii) $\displaystyle\int_1^4(3-x)\sqrt{x}\;\mathrm{d}x=\int_1^4\left(3x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)\mathrm{d}x$

$\displaystyle = \left[3\times\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{2}{5}x^{\frac{5}{2}}\right]_1^4$

$\displaystyle = \left[2x\sqrt{x}-\tfrac{2}{5}x^2\sqrt{x}\right]_1^4$

$\displaystyle = \left(2\times4\times2-\tfrac{2}{5}\times16\times2\right)-\left(2\times1\times1-\tfrac{2}{5}\times1\times1\right)$

$\displaystyle = \tfrac{8}{5}$

Substitute $x = 4$ in first bracket and $x = 1$ in second bracket

Applications of integration

Now that you can integrate a wider range of functions, you can also solve problems which involve integrating these functions, such as finding functions given their gradient function, and finding the area under a curve.

Example 3

The gradient function of a curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{x} - \frac{1}{\sqrt{x}}$$

and the curve passes through the point (4, 9)
Find the equation of the curve.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{x} - \frac{1}{\sqrt{x}} \Rightarrow y = \int\left(3\sqrt{x} - \frac{1}{\sqrt{x}}\right)\mathrm{d}x$$

Integrate to find an expression for y in terms of x

$$= \left(\int 3x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$= 3\times\tfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

$$= 2x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

When $x = 4$, $y = 9$ $\Rightarrow 9 = 2\times 8 - 2\times 2 + c$

Substitute the coordinates of the given point to find the value of the constant c.

$$\Rightarrow c = 9 - 16 + 4$$

$$\Rightarrow c = -3$$

The equation of the curve is $y = 2x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 3$

The next two examples are about finding the area under a curve.

Example 4

Find the area under the graph $y = 1 + \sqrt{x}$ between $x = 0$ and $x = 4$.

Solution

$$\text{Area under graph } = \int_0^4 \left(1 + \sqrt{x}\right)\,\mathrm{d}x$$

$$= \int_0^4 \left(1 + x^{\frac{1}{2}}\right)\,\mathrm{d}x$$

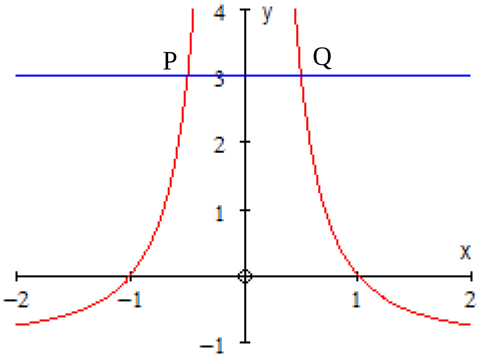
$$= \left[x + \tfrac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \left(4 + \tfrac{2}{3}\times 8\right) - 0$$

$$= \tfrac{28}{3}$$

Example 5

The diagram shows the graph of $y = \frac{1}{x^2} - 1$ and the line $y = 3$.



(i) Find the coordinates of points P and Q.

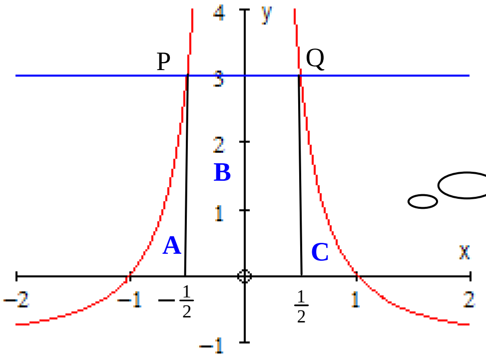
(ii) Find the area bounded by the curve, the line $y = 3$ and the x axis.

Solution

(i) At P and Q, $\frac{1}{x^2} - 1 = 3 \Rightarrow \frac{1}{x^2} = 4 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$.

The coordinates of P are $\left(-\frac{1}{2}, 3\right)$ and the coordinates of Q are $\left(\frac{1}{2}, 3\right)$.

(ii)



> You cannot integrate across a discontinuity. Instead, find the areas of A, B and C separately.

Area C is given by $\int_{\frac{1}{2}}^{1} \left(\frac{1}{x^2} - 1\right) \mathrm{d}x = \left[-\frac{1}{x} - x\right]_{\frac{1}{2}}^{1}$

$= (-1-1) - \left(-2 - \frac{1}{2}\right)$

$= \frac{1}{2}$

By symmetry area A is also $\frac{1}{2}$.

Area B = $3 \times 1 = 3$

Total area = $\frac{1}{2} + \frac{1}{2} + 3 = 4$.

Section 3: Further integration

Exercise level 1

1. Find the following indefinite integrals

(i) $\int \frac{1}{x^2}\,\mathrm{d}x$ (ii) $\int x^{\frac{1}{4}}\,\mathrm{d}x$

(iii) $\int \sqrt[3]{x}\,\mathrm{d}x$ (iv) $\int \left(2x^{\frac{3}{4}} - 3x^{\frac{2}{3}}\right)\mathrm{d}x$

(v) $\int \left(3x^{-3} - 4x^{-4}\right)\mathrm{d}x$ (vi) $\int \left(\frac{1}{x^2} - \frac{2}{x^3}\right)\mathrm{d}x$

2. Evaluate the following definite integrals

(i) $\int_1^3 \frac{1}{x^3}\,\mathrm{d}x$

(ii) $\int_1^9 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

(iii) $\int_1^4 \left(\sqrt{x} - 1\right)\mathrm{d}x$

(iv) $\int_1^3 \frac{1}{x^2} - \frac{1}{x^3}\,\mathrm{d}x$

3. A curve has gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{x} - 3x$ and passes through the point (1, -1). Find the equation of the curve.

4. Find the area under the graph $y = \frac{1}{x^2} + x$ between $x = 1$ and $x = 4$.

Section 3: Further integration

Solutions to Exercise level 1

1. (i) $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx$

$= -x^{-1} + c$

$= -\frac{1}{x} + c$

(ii) $\int x^{\frac{1}{4}}\,dx = \frac{4}{5}x^{\frac{5}{4}} + c$

(iii) $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx$

$= \frac{3}{4}x^{\frac{4}{3}} + c$

(iv) $\int \left(2x^{\frac{3}{4}} - 3x^{\frac{2}{3}}\right)dx = \frac{8}{7}x^{\frac{7}{4}} - \frac{9}{5}x^{\frac{5}{3}} + c$

(v) $\int \left(3x^{-3} - 4x^{-4}\right)dx = -\frac{3}{2}x^{-2} + \frac{4}{3}x^{-3} + c$

(vi) $\int \left(\frac{1}{x^2} - \frac{2}{x^3}\right)dx = -\frac{1}{x} + \frac{1}{x^2} + c$

$= \frac{1-x}{x^2} + c$

2. (i) $\int_1^3 \frac{1}{x^3}\,dx = \int_1^3 x^{-3}\,dx$

$= \left[-\frac{1}{2}x^{-2}\right]_1^3$

$= -\frac{1}{2}(3)^{-2} - \left(-\frac{1}{2}(1)^{-2}\right)$

$= -\frac{1}{18} + \frac{1}{2}$

$= \frac{4}{9}$

(ii) $\int_1^9 \frac{1}{\sqrt{x}}\,dx = \int_1^9 x^{-\frac{1}{2}}\,dx$

$= \left[2x^{\frac{1}{2}}\right]_1^9$

$= \left(2(9)^{\frac{1}{2}}\right) - \left(2(1)^{\frac{1}{2}}\right)$

$= 6 - 2$

$= 4$

(iii) $\int_1^4 \left(\sqrt{x}-1\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - x\right]_1^4$

$= \left(\frac{16}{3}-4\right) - \left(\frac{2}{3}-1\right)$

$= \frac{5}{3}$

(iv) $\int_1^3 \frac{1}{x^2} - \frac{1}{x^3}\,dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_1^3$

$= \left(-\frac{1}{3}+\frac{1}{18}\right) - \left(-1+\frac{1}{2}\right)$

$= \frac{2}{9}$

3. $\frac{dy}{dx} = 2\sqrt{x} - 3x = 2x^{\frac{1}{2}} - 3x$

$y = \int \left(2x^{\frac{1}{2}} - 3x\right)\,dx$

$y = \frac{4}{3}x^{\frac{3}{2}} - \frac{3}{2}x^2 + c$

When $x = 1$, $y = -1$

$-1 = \frac{4}{3}(1)^{\frac{3}{2}} - \frac{3}{2}(1)^2 + c$

$-1 = \frac{4}{3} - \frac{3}{2} + c$

$-1 = -\frac{1}{6} + c$

$c = -\frac{5}{6}$

$y = \frac{4}{3}x^{\frac{3}{2}} - \frac{3}{2}x^2 - \frac{5}{6}$

4. $\int_1^4 \left(\frac{1}{x^2} + x\right)\,dx = \int_1^4 \left(x^{-2} + x\right)\,dx$

$= \left[-x^{-1} + \frac{1}{2}x^2\right]_1^4$

$= \left(-(4)^{-1} + \frac{1}{2}(4)^2\right) - \left(-(1)^{-1} + \frac{1}{2}(1)^2\right)$

$= \left(-\frac{1}{4} + 8\right) - \left(-1 + \frac{1}{2}\right)$

$= \frac{31}{4} + \frac{1}{2}$

$= \frac{33}{4}$

Section 3: Further integration

Exercise level 2

1. Find the following indefinite integrals

(i) $\int \left(3\sqrt{x}-2\right)^2 \mathrm{d}x$ (ii) $\int \frac{(x-1)^2}{x^5}\, \mathrm{d}x$

(iii) $\int \frac{\sqrt{x}+x^{\frac{3}{2}}+7x^3}{\sqrt{x}}\, \mathrm{d}x$ (iv) $\int \frac{(x^2-1)(x^2+1)}{x^2}\, \mathrm{d}x$

2. Evaluate the following definite integrals:

(i) $\int_1^4 \frac{2-x+3x^2}{\sqrt{x}}\, \mathrm{d}x$

(ii) $\int_1^2 \frac{x^2-1}{x^4}\, \mathrm{d}x$

3. A curve has gradient function $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-3}{x^3}$ and passes through the point (1, 1). Find the equation of the curve.

4. Find the area enclosed by the curve $y=x-\sqrt{x}$ and the x-axis.

Section 3: Further integration

Solutions to Exercise level 2

1. (i) $\int (3\sqrt{x}-2)^2\,dx = \int (3x^{\frac{1}{2}}-2)^2\,dx$

$= \int (9x - 12x^{\frac{1}{2}} + 4)\,dx$

$= \frac{9}{2}x^2 - 8x^{\frac{3}{2}} + 4x + c$

(ii) $\int \left(\frac{(x-1)^2}{x^5}\right) dx = \int (x^{-3} - 2x^{-4} + x^{-5})\,dx$

$= -\frac{1}{2}x^{-2} + \frac{2}{3}x^{-3} - \frac{1}{4}x^{-4} + c$

(iii) $\int \frac{\sqrt{x} + x^{\frac{3}{2}} + 7x^3}{\sqrt{x}}\,dx = \int \left(1 + x + 7x^{\frac{5}{2}}\right) dx$

$= x + \frac{1}{2}x^2 + 2x^{\frac{7}{2}} + c$

(iv) $\int \frac{(x^2-1)(x^2+1)}{x^2}\,dx = \int \frac{x^4-1}{x^2}\,dx$

$= \int \left(x^2 - \frac{1}{x^2}\right) dx$

$= \frac{1}{3}x^3 + \frac{1}{x} + c$

2. (i) $\int_1^4 \left(\frac{2 - x + 3x^2}{\sqrt{x}}\right) dx = \int_1^4 \left(2x^{-\frac{1}{2}} - x^{\frac{1}{2}} + 3x^{\frac{3}{2}}\right) dx$

$= \left[4x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}}\right]_1^4$

$= \left(4(4)^{\frac{1}{2}} - \frac{2}{3}(4)^{\frac{3}{2}} + \frac{6}{5}(4)^{\frac{5}{2}}\right) - \left(4(1)^{\frac{1}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} + \frac{6}{5}(1)^{\frac{5}{2}}\right)$

$= \left(8 - \frac{16}{3} + \frac{192}{5}\right) - \left(4 - \frac{2}{3} + \frac{6}{5}\right)$

$= \frac{616}{15} - \frac{68}{15}$

$= \frac{548}{15}$

(ii) $\int_1^2 \left(\frac{x^2-1}{x^4}\right) dx = \int_1^2 \left((x^2-1)(x^{-4})\right) dx$

$= \int_1^2 (x^{-2} - x^{-4})\,dx$

$= \left[-x^{-1} + \frac{1}{3}x^{-3}\right]_1^2$

$$= \left(-(2)^{-1} + \tfrac{1}{3}(2)^{-3}\right) - \left(-(1)^{-1} + \tfrac{1}{3}(1)^{-3}\right)$$
$$= \left(-\tfrac{1}{2} + \tfrac{1}{24}\right) - \left(-1 + \tfrac{1}{3}\right)$$
$$= -\tfrac{11}{24} - \left(-\tfrac{2}{3}\right)$$
$$= \tfrac{5}{24}$$

3. $\dfrac{dy}{dx} = \dfrac{x-3}{x^3} = x^{-2} - 3x^{-3}$

$$y = \int \left(x^{-2} - 3x^{-3}\right) dx$$
$$y = -x^{-1} + \tfrac{3}{2}x^{-2} + c$$

When $x = 1$, $y = 1$

$$1 = -(1)^{-1} + \tfrac{3}{2}(1)^{-2} + c$$
$$1 = -1 + \tfrac{3}{2} + c$$
$$c = \tfrac{1}{2}$$
$$y = -\frac{1}{x} + \frac{3}{2x^2} + \frac{1}{2}$$

4. $y = x - \sqrt{x}$

The graph meets the x-axis when $y = 0$.

$$y = x(1 - x^{-\frac{1}{2}})$$
$$1 - x^{-\frac{1}{2}} = 0 \qquad \text{or} \qquad x = 0$$
$$1 = x^{-\frac{1}{2}}$$
$$x = 1$$

$$\int_0^1 \left(x - x^{\frac{1}{2}}\right) dx = \left[\tfrac{1}{2}x^2 - \tfrac{2}{3}x^{\frac{3}{2}}\right]_0^1$$
$$= \left(\tfrac{1}{2}(1)^2 - \tfrac{2}{3}(1)^{\frac{3}{2}}\right) - \left(\tfrac{1}{2}(0)^2 - \tfrac{2}{3}(0)^{\frac{3}{2}}\right)$$
$$= \left(\tfrac{1}{2} - \tfrac{2}{3}\right) - 0$$
$$= -\tfrac{1}{6}$$

Area cannot be negative so area $= \tfrac{1}{6}$

Topic assessment

1. Find the following indefinite integrals

   (i) $\int (x^4 + 2x^2 - 3)\,\mathrm{d}x$ [2]

   (ii) $\int \left(2\sqrt{x} - \frac{3}{\sqrt{x}}\right)\mathrm{d}x$ [3]

   (iii) $\int \left(\frac{2}{x^2} - \frac{1}{2x^3}\right)\mathrm{d}x$ [3]

2. Find the following definite integrals

   (i) $\int_{-1}^{2} x^2\,\mathrm{d}x$ [2]

   (ii) $\int_{-2}^{-1} (1+2x^3)\,\mathrm{d}x$ [3]

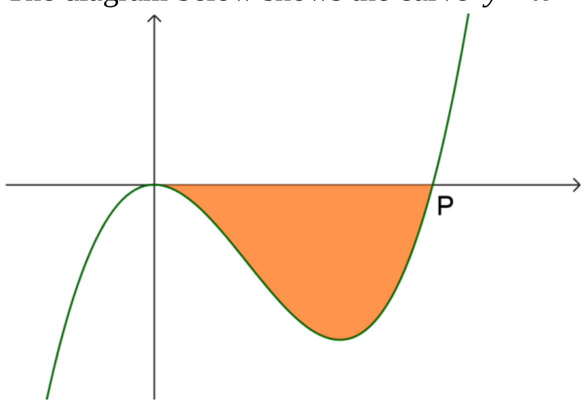
   (iii) $\int_{1}^{9} (2-x)\sqrt{x}\,\mathrm{d}x$ [4]

   (iv) $\int_{1}^{2} \left(\frac{6}{x^2} - \frac{8}{x^3}\right)\mathrm{d}x$ [4]

3. Find the equation of the graph with gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 2x + 3$ which passes through the point (1, 2). [4]

4. Find the equation of the graph with gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2} - x\sqrt{x}$ which passes through the point (1, 2). [4]

5. The diagram below shows the curve $y = x^3 - 2x^2$.



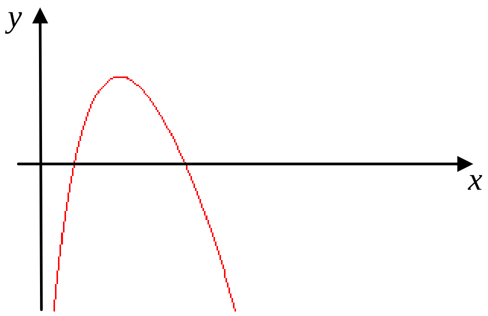
   (i) Find the coordinates of the point P. [2]

   (ii) Find the shaded area. [5]

6. (i) Sketch the curve $y = 2 - x - x^2$ [2]

   (ii) Find the area enclosed by the curve and the x-axis. [5]

7. The diagram shows part of the graph of $y = 5 - x^2 - \frac{4}{x^2}$.



   Find the area enclosed between the graph and the x-axis. [7]

Total: 50 marks

Summary sheet: Integration

H1 Know and use the Fundamental Theorem of Calculus
H2 Integrate x^n (excluding n = -1), and related sums, differences and constant multiples
H3 Evaluate definite integrals; use a definite integral to find the area under a curve

Notation

Integration questions are usually given as follows:

$$\int (3x^2 + 7x - 1)\,\mathrm{d}x$$

Means: *Integrate the following* ($\int$) *with respect to x* ($\mathrm{d}x$)

The fundamental theorem of calculus

Remember that integration is the reverse of differentiation (they 'undo' each other). The rule for differentiation is: expression X power, then reduce the power by 1. So integration is the opposite:

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$

and

$$\int kx^n \,\mathrm{d}x = \frac{kx^{n+1}}{n+1} + c$$

Add 1 to the power then divide the expression by the <u>new</u> power

Remember for each term: **power +1** then **divide by new power**.

e.g. $\displaystyle\int (\mathbf{15x^2 - 6x + 7})\,\mathbf{d}x$

$$= \frac{15x^3}{3} - \frac{6x^2}{2} + 7x + c$$

$$= 5x^3 - 3x^2 + 7x + c$$

Remember to include the c because there could have been a number in the original that disappeared when differentiating.

You will only be able to find c if you are given some more information.

e.g. for the above example, when $x = 1, y = 6$. Find the value of c

You have found that $y = 5x^3 - 3x^2 + 7x + c$, so just substitute the given values in to find c.

$$6 = 5(1)^3 - 3(1)^2 + 7(1) + c$$
$$6 - 5 + 3 - 7 = c$$
$$c = -3$$

So the final answer is: $y = 5x^3 - 3x^2 + 7x - 3$

Summary sheet: Integration

Definite integrals

A definite integral has limits. To evaluate a definite integral you integrate as normal then substitute the top limit and the bottom limit and subtract.

$$[\boldsymbol{top\ limit}] - [\boldsymbol{bottom\ limit}]$$

Remember that definite integration is used to find the area under a curve ("Under the curve" means between the curve and the x-axis).

e.g. Find the area enclosed by the curve $y = -x^2 + 7x - 10$ and the lines $x = 3$ and $x = 5$

Upper limit (5)

Set up the integration: $$\int_3^5 (-x^2 + 7x - 10)\mathrm{d}x$$

Lower limit (3)

Integrate: $$= \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_3^5$$

Notice that you don't need to include the c, because you are going to subtract, so it would cancel out anyway.

$[top\ limit] - [bottom\ limit]$ $$= \left[-\frac{5^3}{3} + \frac{7(5)^2}{2} - 10(5)\right] - \left[-\frac{3^3}{3} + \frac{7(3)^2}{2} - 10(3)\right]$$

$$= \left[-\frac{25}{6}\right] - \left[-\frac{15}{2}\right]$$

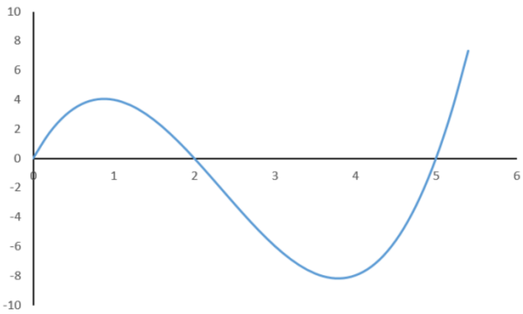
$$= \frac{10}{3}\ (3.3)$$

You have found the area under the curve, between $x = 3$ and $x = 5$.

> **Remember**:
> A **positive** answer means that the area is **above** the x-axis and a **negative** answer means that the area is **below** the x-axis.

If there is a mixture (above and below) you would need to find each area separately and then add the areas (ignoring the negative sign).

e.g. to find the area enclosed by the curve $y = x^3 - 7x^2 + 10x$ and the x-axis:



You would integrate with the limits 0 and 2 then **separately** integrate with the limits 2 and 5 (expect a negative answer as this area is below the line).

Total area: ignore the negative sign and add the 2 amounts together.

Try it – you should get an area of $\frac{253}{12}$ (approx. 21.1)

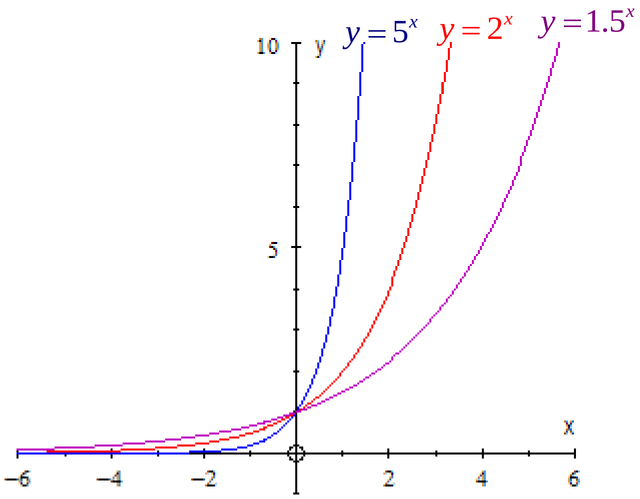
Section 1: Exponential functions and logarithms

Notes and Examples

These notes contain subsections on

Exponential functions

An exponential function is any function of the form $y = a^x$. The graphs below show some different exponential functions.

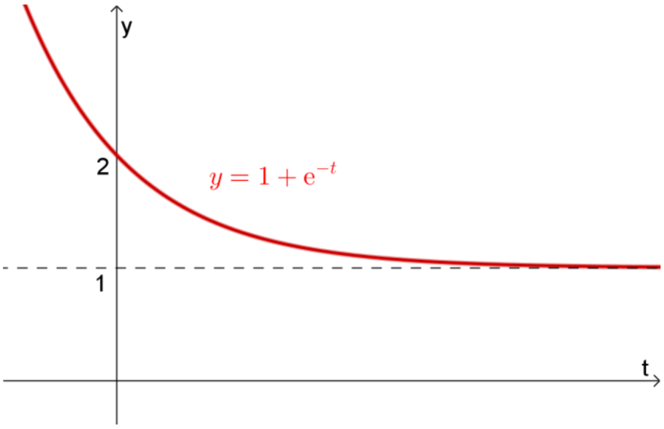


Many real life situations can be modelled by exponential functions. The growth of a population (e.g. of people, animals or bacteria) can be modelled by an exponential function. A model like this might take the form $y = c \times a^{kt}$. This type of model is called **exponential growth**.

In an exponential growth model, the quantity being modelled continues to increase, at an ever-increasing rate. In a real-life situation such as the growth of a population, the model will eventually break down, since other factors such as overcrowding or limited resources will affect the growth of the population.

Another type of model is **exponential decay**, in which something decreases exponentially. A model like this might take the form $y = c \times a^{-kt}$. Exponential decay could model the temperature of a cooling liquid, or the mass of a radioactive isotope remaining.

In an exponential decay model, the quantity being modelled decreases at a rate which becomes slower and slower. The quantity will approach a limiting value, but never quite reach it. For example, the graph below shows the curve $y = 1 + e^{-t}$. The graph approaches the line $y = 1$ as t becomes large.



Indices and logarithms

Logarithms are the inverse of exponentials.

The important thing to remember about logarithms is that, although they appear to be a new topic, they are simply about writing what you already know about indices in a different way.

If you find it difficult to work out the meaning of a statement involving logarithms, it can be simpler to change the statement into the equivalent statement involving indices.

$$\log_a b = x \Leftrightarrow a^x = b .$$

To remember this, notice that a is both the base of the logarithm and the base of the index, and x, the logarithm, is the index. The value of $\log_a b$ is the answer to the question: “What power must I raise a to in order to get b?”

Example 1

(i) Find $\log_4 2$

(ii) Find x, where $\log_5 x = -\frac{1}{2}$

Solution

(i) The statement $\log_4 2 = x$ is equivalent to $4^x = 2$.

Since $4^{\frac{1}{2}} = 2$, then x must be $\dfrac{1}{2}$.

So $\log_4 2 = \frac{1}{2}$.

(ii) The statement $\log_5 x = -\frac{1}{2}$ is equivalent to $5^{-\frac{1}{2}} = x$.

So $x = \dfrac{1}{\sqrt{5}}$.

Try the ***Introduction to logarithms walkthrough*** and the ***Introduction to logarithms skill pack***.

The laws of logarithms

The laws of logarithms are:

$$\log x + \log y = \log xy$$

$$\log x - \log y = \log \frac{x}{y}$$

$$\log x^n = n \log x$$

It's a worthwhile exercise to try to work through these proofs

These can be proved using the laws of indices:

First convert into index notation: $\log_c x = a \Leftrightarrow c^a = x$

$$\log_c y = b \Leftrightarrow c^b = y$$

To prove the first law:

Using the laws of indices

$$c^a c^b = xy \Leftrightarrow c^{a+b} = xy$$

$$\Leftrightarrow \log_c xy = a + b$$

$$\Leftrightarrow \log_c xy = \log_c x + \log_c y$$

Similarly for the second law:

$$\frac{c^a}{c^b} = \frac{x}{y} \Leftrightarrow c^{a-b} = \frac{x}{y}$$

$$\Leftrightarrow \log_c \frac{x}{y} = a - b$$

$$\Leftrightarrow \log_c \frac{x}{y} = \log_c x - \log_c y$$

For the third law:

$$\log_c x^n = a \Leftrightarrow c^a = x^n$$

$$\Leftrightarrow c^{a/n} = x$$

$$\Leftrightarrow \log_c x = \frac{a}{n}$$

$$\Leftrightarrow n \log_c x = a$$

$$\Leftrightarrow n \log_c x = \log_c x^n$$

As the first two laws of indices require the indices to have the same base, then the first two laws of logarithms require the logarithms to have the same base.

Example 2

(i) Write $\log \dfrac{x^3 y}{\sqrt{z}}$ in terms of log *x*, log *y* and log *z*.

(ii) Write $2\log a - \log b - \frac{1}{3}\log c$ as a single logarithm.

Solution

(i)
$$\begin{aligned}\log \frac{x^3 y}{\sqrt{z}} &= \log x^3 + \log y - \log \sqrt{z} \\ &= \log x^3 + \log y - \log z^{1/2} \\ &= 3\log x + \log y - \tfrac{1}{2}\log z\end{aligned}$$

(ii)
$$\begin{aligned}2\log a - \log b - \tfrac{1}{3}\log c &= \log a^2 - \log b - \log c^{1/3} \\ &= \log a^2 - (\log b + \log \sqrt[3]{c}) \\ &= \log \frac{a^2}{b\sqrt[3]{c}}\end{aligned}$$

Try the ***Properties of logarithms walkthrough*** and the ***Laws of logarithms skill pack***.

Solving exponential equations using logarithms

Many equations are solved using inverse functions, for example you solve the equation $x + 3 = 5$, in which addition is applied to the unknown variable, by subtracting 3 from each side. Similarly you solve the equation $x^2 = 10$ by using the square root function, which is the inverse of the square function.

Exponential functions are the inverse of logarithm functions: the function $y = a^x$ is the inverse of the function $y = \log_a x$. An equation like $2^x = 10$ involves an exponential function of *x*. So to solve this equation, it follows that you need to use the inverse of the exponential function, which is the logarithm function. This is shown in the next example.

Example 3

Solve the following equations.

(i) $2^x = 10$

(ii) $3^{2x-1} = 4$

(iii) $0.2^{1-x} = 2$

Solution

(i) $2^x = 10$

$\log 2^x = \log 10$

$x \log 2 = \log 10$

$x = \dfrac{\log 10}{\log 2} = 3.32$

(ii) $3^{2x-1} = 4$

$\log 3^{2x-1} = \log 4$

$(2x-1)\log 3 = \log 4$

$2x - 1 = \dfrac{\log 4}{\log 3}$

$x = \dfrac{1}{2}\left(\dfrac{\log 4}{\log 3} + 1\right) = 1.13$

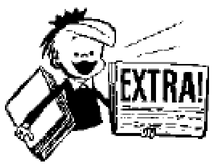
(iii) $0.2^{1-x} = 2$

$\log 0.2^{1-x} = \log 2$

$(1-x)\log 0.2 = \log 2$

$1 - x = \dfrac{\log 2}{\log 0.2}$

$x = 1 - \dfrac{\log 2}{\log 0.2} = 1.43$



An old practical application of logarithms

Before calculators existed, logarithms were used to make calculations easier. For example, suppose you had to divide 1432627 by 967253. You could do this by long division, but it would take a long time and the chances of making a mistake would be quite high. So you would apply the second law of logarithms:

$$\log (1432627 \div 967253) = \log 1432627 - \log 967253$$

To do the calculation, you would have to find the log to base 10 of the two numbers, subtract the results, and then find the inverse log of the answer.

You would have to find the values of $\log 1432627$ and $\log 967253$ from a book of tables. Unfortunately most tables would only tell you the values of $\log x$ for values of x between 10 and 99. So you would then use the fact that

$$\begin{aligned}\log\ 1432627 &= \log(14.32627 \times 100000)\\ &= \log 14.32627 + \log 100000\\ &= \log 14.32627 + 5\end{aligned}$$

You would then use the tables to find the value of $\log 14.33$ (which is as accurate as most tables would give you). This would give a value for $\log 1432627$ of 6.1562.

You would then go through a similar process to find $\log 967253$.

$$\begin{aligned}\log 967253 = \log 96.7253 + \log 10000\\ &= 1.9855 + 4\\ &= 5.9855\end{aligned}$$

Next you would subtract these two logarithms (without a calculator of course!), giving 0.1707.

Now you would have to find the number whose logarithm is 0.1707. Inverse log tables usually give values between 1 and 10.

$$\begin{aligned}0.1707 &= 1.1707 - 1\\ &= \log\ 14.81 - \log\ 10\\ &= \log \frac{14.81}{10}\\ &= \log 1.481\end{aligned}$$

So $1432627 \div 967253 = 1.481$.

Most pupils did not understand the theory behind these calculations; they just followed a set of instructions to use the tables of logarithms and work out the calculation. Even after calculators became widely available, it was several years before this technique was removed from examination syllabuses!

Logarithms were also the basis of slide rules, which were also used before calculators existed to work out calculations quickly.

Section 1: Exponential functions and logarithms

Exercise level 1

1. Rewrite each of these statements as a logarithm.

   (i) $10^3 = 1000$ (ii) $2^7 = 128$

   (iii) $10^{\frac{1}{3}} = \sqrt[3]{10}$ (iv) $2^{-3} = \frac{1}{8}$

   (v) $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$ (vi) $3^{\frac{3}{2}} = \sqrt{27}$

2. Find the values of the following:

   (i) $\log_2 16$ (ii) $\log_{10} 1000000$

   (iii) $\log_6 1$ (iv) $\log_4 \left(\frac{1}{4}\right)$

   (v) $\log_5 \sqrt{5}$ (vi) $\log_3 \left(\frac{1}{27}\right)$

   (vii) $\log_8 4$ (viii) $\log_2 \left(\frac{1}{\sqrt{32}}\right)$

3. Find the value of x in each of the following:

   (i) $\log_2 x = -5$ (ii) $\log_3 x = \frac{3}{2}$

   (iii) $\log_x 64 = 2$ (iv) $\log_x \left(\frac{1}{\sqrt{5}}\right) = \frac{1}{2}$

4. Write the following as a single logarithm:

   (i) $\log 2 + \log 3$ (ii) $\log 10 - \log 2$

   (iii) $3 \log 5$ (iv) $2 \log 3 - 4 \log 2$

   (v) $\frac{1}{2}\log 3 - \frac{1}{4}\log 4$ (vi) $2 \log a + 5 \log b - 3 \log c$

5. Write as a single expression:

   (i) $\frac{1}{2}\log 2 - \frac{1}{4}\log 16$

   (ii) $4\log_{10} 3 - \log_{10} 9$

   (iii) $4\log_3 3 - \log_3 9$

   (iv) $2\log x + 3\log y - \log(x^2 y)$

   (v) $a\log b - b\log a + \log a - \log b$

Section 1: Exponential functions and logarithms

Solutions to Exercise level 1

1. (i) $10^3 = 1000 \quad \Rightarrow \log_{10} 1000 = 3$

   (ii) $2^7 = 128 \quad \Rightarrow \log_2 128 = 7$

   (iii) $10^{\frac{1}{3}} = \sqrt[3]{10} \quad \Rightarrow \log_{10} \sqrt[3]{10} = \frac{1}{3}$

   (iv) $2^{-3} = \frac{1}{8} \quad \Rightarrow \log_2 \frac{1}{8} = -3$

   (v) $5^{-\frac{1}{2}} = \dfrac{1}{\sqrt{5}} \quad \Rightarrow \log_5 \dfrac{1}{\sqrt{5}} = -\frac{1}{2}$

   (vi) $3^{\frac{3}{2}} = \sqrt{27} \quad \Rightarrow \log_3 \sqrt{27} = \frac{3}{2}$

2. (i) $x = \log_2 16 \quad \Rightarrow 2^x = 16 \quad \Rightarrow x = 4$
   so $\log_2 16 = 4$

   (ii) $x = \log_{10} 1000000 \quad \Rightarrow 10^x = 1000000 \quad \Rightarrow x = 6$
   so $\log_{10} 1000000 = 6$

   (iii) $x = \log_6 1 \quad \Rightarrow 6^x = 1 \quad \Rightarrow x = 0$
   so $\log_6 1 = 0$

   (iv) $x = \log_4 \left(\dfrac{1}{4}\right) \quad \Rightarrow 4^x = \frac{1}{4} \quad \Rightarrow x = -1$
   so $\log_4 \left(\dfrac{1}{4}\right) = -1$

   (v) $x = \log_5 \sqrt{5} \quad \Rightarrow 5^x = \sqrt{5} \quad \Rightarrow x = \frac{1}{2}$
   so $\log_5 \sqrt{5} = \frac{1}{2}$

   (vi) $x = \log_3 \left(\dfrac{1}{27}\right) \quad \Rightarrow 3^x = \frac{1}{27} \quad \Rightarrow x = -3$
   so $\log_3 \left(\dfrac{1}{27}\right) = -3$

   (vii) $x = \log_8 4 \quad \Rightarrow 8^x = 4 \quad \Rightarrow 2^{3x} = 2^2 \quad \Rightarrow x = \frac{2}{3}$
   so $\log_8 4 = \frac{2}{3}$

(viii) $x = \log_2\left(\frac{1}{\sqrt{32}}\right) \quad \Rightarrow 2^x = \frac{1}{\sqrt{32}} = \frac{1}{\sqrt{2^5}} = 2^{-\frac{5}{2}} \quad \Rightarrow x = -\frac{5}{2}$

so $\log_2\left(\frac{1}{\sqrt{32}}\right) = -\frac{5}{2}$

3. (i) $\log_2 x = -5 \quad \Rightarrow 2^{-5} = x \quad \Rightarrow x = \frac{1}{32}$

(ii) $\log_3 x = \frac{3}{2} \quad \Rightarrow 3^{\frac{3}{2}} = x \quad \Rightarrow x = \sqrt{27}$

(iii) $\log_x 64 = 2 \quad \Rightarrow x^2 = 64 \quad \Rightarrow x = 8$

(iv) $\log_x\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{2} \quad \Rightarrow x^{\frac{1}{2}} = \frac{1}{\sqrt{5}} \quad \Rightarrow x = \frac{1}{5}$

4. (i) $\log 2 + \log 3 = \log(2 \times 3) = \log 6$

(ii) $\log 10 - \log 2 = \log \frac{10}{2} = \log 5$

(iii) $3\log 5 = \log 5^3 = \log 125$

(iv) $2\log 3 - 4\log 2 = \log 3^2 - \log 2^4 = \log \frac{3^2}{2^4} = \log \frac{9}{16}$

(v) $\frac{1}{2}\log 3 - \frac{1}{4}\log 4 = \log 3^{\frac{1}{2}} - \log 4^{\frac{1}{4}} = \log \frac{\sqrt{3}}{\sqrt{2}} = \log \sqrt{\frac{3}{2}}$

(vi) $2\log a + 5\log b - 3\log c = \log a^2 + \log b^5 - \log c^3 = \log \frac{a^2 b^5}{c^3}$

5. (i) $\frac{1}{2}\log 2 - \frac{1}{4}\log 16 = \log 2^{\frac{1}{2}} - \log 16^{\frac{1}{4}}$

$$= \log\sqrt{2} - \log 2$$

$$= \log\left(\frac{\sqrt{2}}{2}\right)$$

$$= \log\left(\frac{1}{\sqrt{2}}\right) \quad (\text{or} \quad -\log\sqrt{2}\ )$$

AQA AS Maths Exponentials and logs 1 Exercise solutions

(ii) $$4\log_{10}3-\log_{10}9=\log_{10}3^3-\log_{10}9$$
$$=\log_{10}\left(\frac{3^4}{9}\right)$$
$$=\log_{10}9$$

(iii) $$4\log_3 3-\log_3 9=4\log_3 3-\log_3 3^2$$
$$=4\log_3 3-2\log_3 3$$
$$=4-2=2$$

(iv) $$2\log x+3\log y-\log(x^2y)=\log x^2+\log y^3-\log(x^2y)$$
$$=\log\left(\frac{x^2y^3}{x^2y}\right)$$
$$=\log y^2\ \ (\text{or } 2\log y)$$

(v) $$a\log b-b\log a+\log a-\log b=(1-b)\log a+(a-1)\log b$$
$$=\log a^{1-b}+\log b^{a-1}$$
$$=\log\left(a^{1-b}b^{a-1}\right)$$

Section 1: Exponential functions and logarithms

Exercise level 2

1. Write the following in terms of log 2 and log 3:

   (i) $\log 12$ (ii) $\log\left(\frac{16}{27}\right)$

   (iii) $\log\sqrt{54}$ (iv) $\log\frac{\sqrt{3}}{16}$

2. Solve the following equations:

   (i) $2^x = 18$ (ii) $5^x = 100$

   (iii) $1.5^x = 0.001$ (iv) $10^x = 2$

3. Solve the equations:

   (i) $\log_3 x = -\frac{1}{2}$

   (ii) $2\log x + \log 10 = \log 90$

   (iii) $\log_{10}\left(\frac{1}{x}\right) = -3$

   (iv) $\log x + \log y = \frac{1}{2}\log\left(9y^2\right)$

4. Money in a savings account earns 4% interest per year. The amount of money in Claire's savings account is modelled as $P = 500\times 1.04^t$, where t is the number of years. Interest is paid at the end of each month.

   (i) What is the total amount, to the nearest penny, in Claire's account after

   (a) 1 month

   (b) 6 months

   (c) 1 year

   (d) 5 years

   (ii) After how long (in years and months) will the amount in the account first exceed £800?

5. Solve the equations:

   (i) $3^a = 21$

   (ii) $(1.005)^x = 1.1$

   (iii) $50^{\frac{1}{a}} = 10$

   (iv) $\left(1+\frac{x}{100}\right)^{12} = 1.25$

6. Interpret the question and answer to 4(iv) above, in the context of the monthly interest rate charged for a debt on a credit card.

Section 1: Exponential functions and logarithms

Solutions to Exercise level 2

1. (i) $\log 12 = \log(2^2 \times 3) = \log 2^2 + \log 3 = 2\log 2 + \log 3$

(ii) $\log\left(\frac{16}{27}\right) = \log\left(\frac{2^4}{3^3}\right) = \log 2^4 - \log 3^3 = 4\log 2 - 3\log 3$

(iii) $\log\sqrt{54} = \log\left(2\times 3^3\right)^{\frac{1}{2}} = \log 2^{\frac{1}{2}} + \log 3^{\frac{3}{2}} = \frac{1}{2}\log 2 + \frac{3}{2}\log 3$

(iv) $\log\frac{\sqrt{3}}{16} = \log\frac{3^{\frac{1}{2}}}{2^4} = \log 3^{\frac{1}{2}} - \log 2^4 = \frac{1}{2}\log 3 - 4\log 2$

2. (i) $2^x = 18$

$\log 2^x = \log 18$

$x\log 2 = \log 18$

$x = \frac{\log 18}{\log 2} = 4.17$ (3 s.f.)

(ii) $5^x = 100$

$\log 5^x = \log 100$

$x\log 5 = \log 100$

$x = \frac{\log 100}{\log 5} = 2.86$ (3 s.f.)

(iii) $1.5^x = 0.001$

$\log 1.5^x = \log 0.001$

$x\log 1.5 = \log 0.001$

$x = \frac{\log 0.001}{\log 1.5} = -17.0$ (3 s.f.)

(iv) $10^x = 2$

$\log 10^x = \log 2$

$x\log 10 = \log 2$

$x = \frac{\log 2}{\log 10} = 0.301$ (3 s.f.)

AQA AS Maths Exponentials and logs 1 Exercise solutions

3. (i) $\log_3 x = -\frac{1}{2}$

$\Rightarrow x = 3^{-\frac{1}{2}} = \frac{1}{\sqrt{3}}$

(ii) $2\log x + \log 10 = \log 90$

$\Rightarrow \log x^2 + \log 10 = \log 90$

$\Rightarrow 10x^2 = 90$

$\Rightarrow x^2 = 9$

$\Rightarrow x = 3$

(Note that x = -3 does not satisfy the equation, as log (-3) is undefined)

(iii) $\log_{10}\left(\frac{1}{x}\right) = -3$

$\Rightarrow \left(\frac{1}{x}\right) = 10^{-3} = \frac{1}{1000}$

$\Rightarrow x = 1000$

(iv) $\log x + \log y = \frac{1}{2}\log\left(9y^2\right)$

$\Rightarrow \log(xy) = \log(3y)$

$\Rightarrow xy = 3y$

$\Rightarrow x = 3$

4. $P = 500 \times 1.04^t$

(i) (a) After 1 month, $t = \frac{1}{12}$.

$P = 500 \times 1.04^{\frac{1}{12}} = 501.64$

Amount after 1 month = £501.64.

(b) After 6 months, $t = 0.5$

$P = 500 \times 1.04^{0.5} = 509.90$

Amount after 6 months = £509.90.

(c) After 1 year, $t = 1$

$P = 500 \times 1.04^{1} = 520$

Amount after 6 months = £520.00.

(d) After 5 years, $t = 5$

$P = 500 \times 1.04^{5} = 608.33$

Amount after 6 months = £608.33.

(ii) $500\times 1.04^{t} > 800$

$1.04^{t} > 1.6$

$t\log 1.04 > \log 1.6$

$t > 11.98$

11.98 years = 11 years 11.76 months

It will first exceed £800 after 12 years 0 months.

5. (i) $3^{a} = 21$

$\Rightarrow a\log 3 = \log 21$

$\Rightarrow a = \dfrac{\log 21}{\log 3} = 2.77$ (3 s.f.)

(ii) $(1.005)^{x} = 1.1$

$\Rightarrow x\log(1.005) = \log(1.1)$

$\Rightarrow x = \dfrac{\log(1.1)}{\log(1.005)} = 19.1$ (3 s.f.)

(iii) $50^{\frac{1}{a}} = 10$

$\Rightarrow \dfrac{1}{a}\log 50 = \log 10$

$\Rightarrow a = \dfrac{\log 50}{\log 10} \approx 1.699$ (3 d.p.)

(iv) $\left(1+\dfrac{x}{100}\right)^{12} = 1.25$

$\Rightarrow 12\log\left(1+\dfrac{x}{100}\right) = \log(1.25)$

$\Rightarrow \log\left(1+\dfrac{x}{100}\right) = \dfrac{1}{12}\log(1.25) = 0.0080758......$

$\Rightarrow 1+\dfrac{x}{100} = 10^{0.0080758} = 1.018769......$

$\Rightarrow x = 1.877$ (3 d.p.)

6. The solution in 4 (iv) shows that a monthly interest rate of approximately 1.877% is equivalent to an annual rate of 25% on a debt.

Section 2: Natural logarithms and exponentials

Notes and Examples

These notes contain subsections on:

Natural logarithms

You have already met logarithms section 1. In this section you are looking at natural logarithms, which are logarithms to base e, where e is a particular irrational number ($e = 2.71828\ldots$. to 5 decimal places). Working with natural logarithms is quite straightforward: they obey exactly the same rules as all other logarithms, so you already know quite a lot about them. Your calculator will be able to work out natural logarithms – you should have a button marked "ln" (note that this is ln not In; some people get confused about this!)

You may be wondering (quite reasonably) what is so special about this number e, and why logarithms to this particular base are so useful. In fact, this number has very many interesting properties, some of which you will learn about in this module and in others:

- Natural logarithms are related to the area under the curve $y = \frac{1}{x}$. You will find out more about this in A level Mathematics.
- One of the interesting characteristics of the exponential function e^x is that the gradient of the graph of $y = e^x$ is equal to the value of e^x at all points.
- In A level Mathematics you will learn to form and solve simple differential equations, which often involve modelling with exponential functions.
- If you go on to study Further Maths 'A' level, you will find that the number e^x can be written as the infinite series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ (try working out the sum of the first 10 or so terms of this series for $x = 1$ on your calculator, and see how close you get to the value of e).

But for now, all that is necessary is to accept that e, like π, is a special and useful number, and that this means that logarithms to base e are also particularly useful.

AQA AS Maths Exp and logs 2 Notes & Examples

Solving equations using natural logarithms and exponentials

Most of the work in this section involves the same techniques as you used in section 1. You need to use the same rules of logarithms, applied to natural logarithms:

$$\ln a + \ln b = \ln ab$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

$$\ln a^n = n \ln a$$

Also remember that since the $\log_a a = 1$ for any value of a, then

$$\ln \mathrm{e} = 1$$

This means that you can use natural logarithms to solve equations involving exponentials, as in Example 1.


Example 1
Solve the equation $\mathrm{e}^{1-3x} = 5$.

Solution

Take natural logarithms of both sides: $\ln \mathrm{e}^{1-3x} = \ln 5$

Using the laws of logarithms: $(1-3x)\ln \mathrm{e} = \ln 5$

Since ln e = 1: $1 - 3x = \ln 5$

$$x = \frac{1-\ln 5}{3} = -0.203$$

Similarly, the relationship between exponentials and logarithms

$$a = \ln b \Leftrightarrow e^a = b$$

allows you to solve equations involving natural logarithms.


Example 2
Solve the equation $\ln(2x+1) = 3$.

Solution

$$\ln(2x+1) = 3 \quad \Rightarrow 2x + 1 = \mathrm{e}^3$$

$$\Rightarrow x = \frac{e^3 - 1}{2} = 9.54$$

In Examples 1 and 2 above, you are using the fact that the exponential function and the natural logarithm function are inverses of one another, in the same way that squaring and the square root function are inverses of one another. (You will learn more about functions and their inverses in chapter 3).

AQA AS Maths Exp and logs 2 Notes & Examples

In Example 1 you are "undoing" an exponential function by using natural logarithms, and in Example 2 you are "undoing" a natural logarithm by using an exponential.

The inverse nature of these two functions can be summed up as follows:

$$\ln e^x = x$$
$$e^{\ln x} = x$$

Exponential functions as models

An exponential function is any function of the form a^x. The function e^x is an example of an exponential function, and is often called "the exponential function".

Many real life situations can be modelled by exponential functions. In section 1 you saw situations which can be modelled by functions like $y = c \times a^{kt}$ (exponential growth) or $y = c \times a^{-kt}$ (exponential decay). Often, functions are modelled using the exponential function e^x, giving rise to models of the form $y = ce^{kt}$ or $y = ce^{-kt}$.

You can solve problems involving exponential growth and decay using logarithms.

Example 3

The temperature T°C of a cup of coffee after t minutes is given by $T = 20 + 60e^{-0.1t}$.

(i) What is the initial temperature of the coffee?

(ii) What is the temperature of the coffee after 5 minutes?

(iii) After how long is the temperature of the coffee 25°C?

(iv) What is the temperature of the room?

Solution

(i) When $t = 0$, $T = 20 + 60 = 80$.
The initial temperature of the coffee is 80°C.

(ii) When $t = 5$, $T = 20 + 60e^{-0.5} = 56.4$
The temperature after 5 minutes is 56.4°.

(iii) When $T = 25$, $25 = 20 + 60e^{-0.1t}$

$$5 = 60e^{-0.1t}$$
$$e^{-0.1t} = \tfrac{1}{12}$$

Taking logarithms: $-0.1t = \ln\tfrac{1}{12}$

$$y = -10\ln\tfrac{1}{12} = 10\ln 12 = 24.8$$

It takes 24.8 minutes.

(iv) As t becomes very large, the temperature approaches a limiting value of 20°C, which is the temperature of the room.

The derivative of the exponential function

As mentioned earlier, one of the interesting characteristics of the function $y = e^x$ is that its gradient is equal to the value of the y-coordinate at every point on the graph.

So $y = e^x \quad \Rightarrow \frac{dy}{dx} = e^x$.

If you replace x with kx, you are stretching the graph parallel to the x-axis with scale factor $\frac{1}{k}$. This has the effect of making the graph k times as steep.

So $y = e^{kx} \quad \Rightarrow \frac{dy}{dx} = ke^{kx}$.

This result allows you to find the rate of change of an exponential function.

Section 2: Natural logarithms and exponentials

Exercise level 1

1. Use your calculator to work out the value of
   (i) e^2 (ii) e^{-3} (iii) $e^{-0.6}$
   (iv) $\ln 2$ (v) $\ln 0.3$ (vi) $\ln 5$

2. Solve each of the following equations
   (i) $e^x = 2$ (ii) $e^{2x-1} = 3$ (iii) $e^x = 2e^{1-2x}$
   (iv) $\ln x = 5$ (v) $\ln x^2 = -2$ (vi) $\ln x = 3 - \ln 2x$

3. Find $\frac{dy}{dx}$ for each of the following:
   (i) $y = e^{2x}$ (ii) $y = e^{-x}$ (iii) $y = 2e^{-3x}$

4. Sketch the graphs of $y = \ln x$ and $y = e^x$ on the same axes.
   What is the geometrical relationship between these two curves?

5. The number of bacteria, N, in a colony at time t, where t is measured in hours, is given by the equation
   $$N = 1000e^{0.2t}.$$
   (i) How many bacteria are there after 2 hours?
   (ii) After how long has the number of bacteria doubled?

6. The temperature T°C of a hot liquid in a cool room after t minutes is given by the equation
   $$T = 18 + 80e^{-0.5t}.$$
   (i) What is the temperature of the liquid initially?
   (ii) Sketch a graph of the temperature of the liquid against time.
   (iii) What is the temperature of the liquid after 10 minutes?
   (iv) After how long is the temperature of the liquid 25°C?
   (v) What do you think the temperature of the room is?

Section 2: Natural logarithms and exponentials

Solutions to Exercise level 1

1. (i) $e^2 = 7.389$ (4 s.f.)
   (ii) $e^{-3} = 0.04979$ (4 s.f.)
   (iii) $e^{-0.6} = 0.5488$ (4 s.f.)
   (iv) $\ln 2 = 0.6931$ (4 s.f.)
   (v) $\ln 0.3 = -1.204$ (4 s.f.)
   (vi) $\ln 5 = 1.609$ (4 s.f.)

2. (i) $e^x = 2$

   $x = \ln 2 = 0.693$ (3 s.f.)

   (ii) $e^{2x-1} = 3$

   $2x - 1 = \ln 3$

   $2x = \ln 3 + 1$

   $x = \frac{1}{2}(\ln 3 + 1) = 1.05$ (3 s.f.)

   (iii) $e^x = 2e^{1-2x}$

   $x = \ln\left(2e^{1-2x}\right) = \ln 2 + \ln e^{1-2x} = \ln 2 + 1 - 2x$

   $3x = \ln 2 + 1$

   $x = \frac{1}{3}(\ln 2 + 1) = 0.564$ (3 s.f.)

   (iv) $\ln x = 5$

   $x = e^5 = 148$ (3 s.f.)

   (v) $\ln x^2 = -2$

   $2\ln x = -2$

   $\ln x = -1$

   $x = e^{-1} = 0.368$ (3 s.f.)

   (vi) $\ln x = 3 - \ln 2x$

   $\ln x + \ln 2x = 3$

   $\ln 2x^2 = 3$

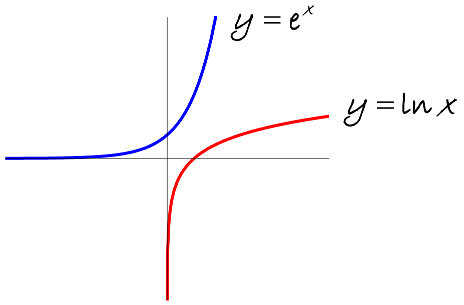
   $2x^2 = e^3$

   $x = \sqrt{\frac{1}{2}e^3} = 3.17$ (3 s.f.)

AQA AS Maths Exp and logs 2 Exercise solutions

3. (i) $\frac{dy}{dx} = 2e^{2x}$ (ii) $\frac{dy}{dx} = -e^{-x}$ (iii) $\frac{dy}{dx} = -6e^{-3x}$

4.



These curves are reflections of each other in the line $y = x$.

5. (i) $N = 1000e^{0.2t}$

When $t = 2$, $N = 1000e^{0.2\times2} = 1000e^{0.4} = 1491.8$

After 2 hours there are 1492 bacteria.

(ii) When $t = 0$, $N = 1000$.

When number has doubled, $1000e^{0.2t} = 2000$

$$e^{0.2t} = 2$$

$$0.2t = \ln 2$$

$$t = 5\ln 2 = 3.47 \text{ (3 s.f.)}$$

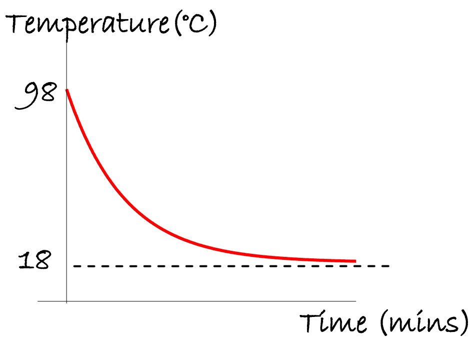
The number of bacteria has doubled after 3.47 hours.

6. (i) $T = 18 + 80e^{-0.5t}$

When $t = 0$, $T = 18 + 80 = 98$

The initial temperature of the liquid is 98°C.

(ii)



(iii) When $t = 10$, $T = 18 + 80e^{-0.5\times10} = 18 + 80e^{-5} = 18.5$

After 10 minutes, the temperature is 18.5°C (3 s.f.)

(iv) When T = 25, $18 + 80e^{-0.5t} = 25$

$$80e^{-0.5t} = 7$$

$$e^{-0.5t} = \tfrac{7}{80}$$

$$-0.5t = \ln\left(\tfrac{7}{80}\right)$$

$$t = -2\ln\left(\tfrac{7}{80}\right) = 4.87$$

The temperature is 25°C after 4.87 minutes.

(v) The temperature of the room is 18°C
(the temperature of the liquid approaches this temperature as t becomes large).

Section 2: Natural logarithms and exponentials

Exercise level 2

1. Make x the subject of $a = be^{-kx}$

2. Make x the subject of $\ln x = a$

3. Make x the subject of $\ln y - \ln x = t$

4. The mass m of a radioactive substance after t seconds is modelled by $m = m_o e^{-kt}$.
The time taken for the mass of the substance to halve is 2 minutes.
(i) Find the value of k to 3 significant figures.
(ii) How long does it take, to the nearest 10 seconds, for the substance to decay to 5% of its original mass?

5. The growth of a population of mice is modelled by $N = 50e^{0.1t}$, where N is the number of mice and t is measured in weeks.
(i) After how many weeks is the number of mice greater than 200?
(ii) What is the rate of increase in the population after 5 weeks?
(iii) Show that $\frac{dN}{dt} = kN$, giving the value of k.
(iv) What is the rate of increase in the population when there are 200 mice?
(v) Explain why this model is unlikely to be appropriate as N becomes very large.

Section 2: Natural logarithms and exponentials

Solutions to Exercise level 2

1. $a = be^{-kx}$ $y = e^x$

$\frac{a}{b} = e^{-kx}$

$\ln\left(\frac{a}{b}\right) = -kx$

$x = -\frac{1}{k}\ln\left(\frac{a}{b}\right) = \frac{1}{k}\ln\left(\frac{b}{a}\right)$

2. $\ln x = a$

$x = e^a$

3. $\ln y - \ln x = t$

$\ln\left(\frac{y}{x}\right) = t$

$\frac{y}{x} = e^t$

$y = xe^t$

$x = ye^{-t}$

4. (i) $m = m_0e^{-kt}$

When $t = 120$, $m = \frac{1}{2}m_0$.

$\frac{1}{2}m_0 = m_0e^{-120k}$

$e^{-120k} = \frac{1}{2}$

$-120k = \ln\frac{1}{2}$

$k = -0.00578$

(ii) $m = m_0e^{-0.00578t}$

When $m = 0.05m_0$, $0.05m_0 = m_0e^{-0.00578t}$

$e^{-0.00578t} = 0.05$

$-0.00578t = \ln 0.05$

$t = 520$ seconds (to nearest 10 seconds)

5. (i) $N = 50e^{0.1t}$

When N = 200

$200 = 50e^{0.1t}$

$e^{0.1t} = 4$

$0.1t = \ln 4$

$t = 13.86$

The population is greater than 200 after 14 weeks.

(ii) $\frac{dN}{dt} = 50 \times 0.1e^{0.1t} = 5e^{0.1t}$

When t = 5, $\frac{dN}{dt} = 5e^{0.5} = 8.24$

Rate of increase after 5 weeks is 8.24 mice / week.

(iii) $N = 50e^{0.1t} \quad \Rightarrow e^{0.1t} = \frac{N}{50}$

$\frac{dN}{dt} = 5 \times \frac{N}{50}$

$\frac{dN}{dt} = \frac{N}{10}$

The value of k is 0.1.

(iv) When N = 200, $\frac{dN}{dt} = \frac{200}{10} = 20$

The rate of increase is 20 mice / week.

(v) Resources such as food and space are unlikely to be able to sustain the population as it becomes very large. Also, the model takes no account of mice dying.

Section 3: Modelling curves

Notes and Examples

These notes contain subsections on:

- Modelling curves of the form $y = kx^n$
- Modelling curves of the form $y = ka^x$

Modelling curves of the form $y = kx^n$

When you collect data from an experiment, you may want to find a relationship between two variables, such as the speed of a moving object at a particular time, or temperature of an object and its distance from a heat source. You may plot a graph of one variable against another to help find this relationship. However, unless the graph is a straight line, it may be difficult to see the relationship from the graph. The graphs of $y = x^2$, $y = x^3$ etc. look quite similar for $x \geq 0$, and as the experimental data may not be very accurate, it can be impossible to tell with any certainty what would be the best graph to model the data.

This is where logarithms can be very useful. If the relationship is of the form $y = kx^n$, then plotting $\log y$ against $\log x$ gives a straight line graph. (Note that you can use logs to any base for this – it is usual to use either logs to base 10, or natural logarithms, since both of these are easily found on a calculator).

$$y = kx^n$$
$$\Rightarrow \log y = \log kx^n$$
$$\Rightarrow \log y = \log k + \log x^n$$
$$\Rightarrow \log y = \log k + n\log x$$
$$\Rightarrow \log y = n\log x + \log k$$

This is of the form $y = mx + c$, so plotting $\log y$ against $\log x$ gives a straight line with gradient n and intercept $\log k$.

The value of n is therefore the gradient of the graph, and the value of k is found by taking the intercept of the graph and finding its inverse logarithm (i.e. $10^{\text{intercept}}$ if you are using logs to base 10, or $e^{\text{intercept}}$ if you are using natural logarithms).

Example 1

The relationship between two variables x and y is believed to be of the form $y = kx^n$, where k and n are constants.

In an experiment, the following values of x and y are recorded.

x	1	2	3	4	5	6	7	8
y	1.98	1.39	1.16	1.01	0.91	0.82	0.75	0.72

Verify that the model $y = kx^n$ is appropriate and find the approximate values of the constants k and n.

Solution

$$y = kx^n$$

Taking logarithms: $\log y = \log kx^n$

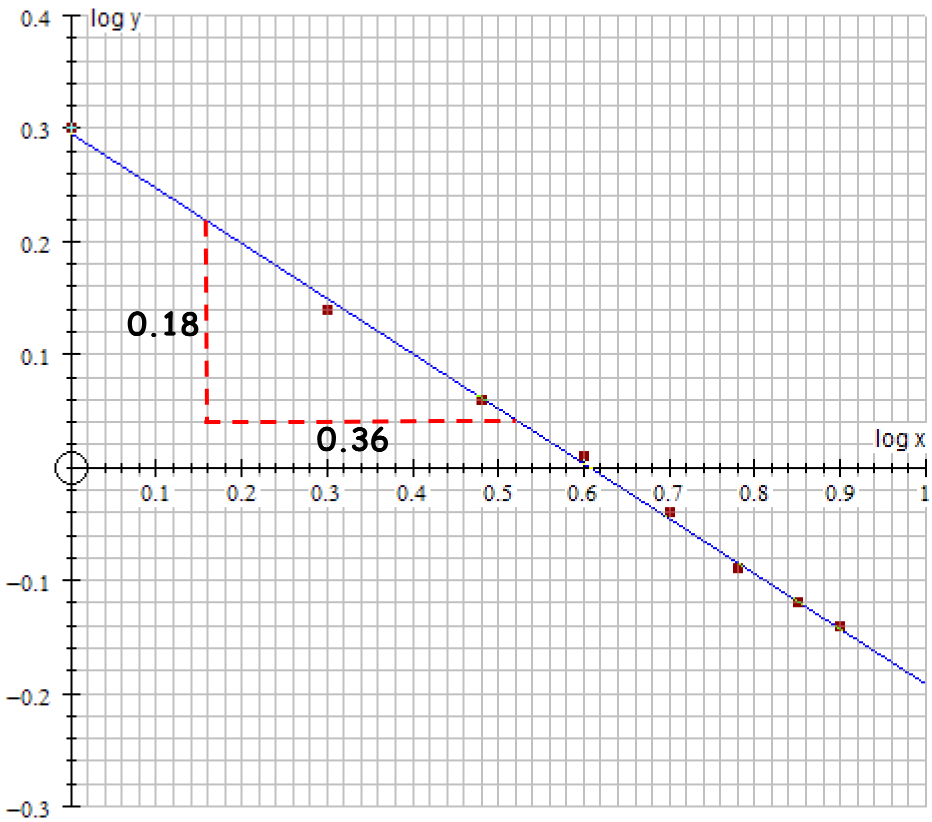
$$\log y = \log k + \log x^n$$

$$\log y = \log k + n \log x$$

If log y is plotted against log x, this is the equation of a straight line graph with gradient n and intercept log k.
Plot the values of log y against log x:

Logs to base 10 have been used here but natural logarithms could be used instead

x	1	2	3	4	5	6	7	8
log x	0	0.30	0.48	0.60	0.70	0.78	0.85	0.90
y	1.98	1.39	1.16	1.02	0.91	0.82	0.75	0.72
log y	0.30	0.14	0.06	0.01	-0.04	-0.09	-0.12	-0.14



Since the graph is approximately a straight line, the relationship $y = kx^n$ is an appropriate model.

Gradient = $n = -\dfrac{0.18}{0.36} = -0.5$

Intercept $= \log k = 0.3 \Rightarrow k = 10^{0.3} \approx 2$

The relationship is approximately $y = 2x^{-0.5} = \frac{2}{\sqrt{x}}$

Modelling curves of the form $y = ka^x$

Similarly, if the relationship is of the exponential form $y = ka^x$, then plotting $\log y$ against x gives a straight line graph.

$$y = ka^x$$
$$\Rightarrow \log y = \log ka^x$$
$$\Rightarrow \log y = \log k + \log a^x$$
$$\Rightarrow \log y = \log k + x\log a$$
$$\Rightarrow \log y = (\log a)x + \log k$$

This is of the form $y = mx + c$, so plotting $\log y$ against x gives a straight line with gradient $\log a$ and intercept $\log k$.

The value of a is found by taking the gradient of the graph and finding its inverse logarithm (i.e. 10^{gradient} if you are using logs to base 10 or e^{gradient} if you are using natural logs), and the value of k is found by taking the intercept of the graph and finding its inverse logarithm (i.e. $10^{\text{intercept}}$ if you are using logs to base 10 or $e^{\text{intercept}}$ if you are using natural logs).

Example 2

The relationship between two variables p and q is believed to be of the form $q = ab^p$, where a and b are constants.

In an experiment, the following values of p and q are recorded.

p	1.5	2.0	2.5	3.0	3.5	4.0
q	12	19	30	46	74	116

Verify that the model $q = ab^p$ is appropriate, and estimate the values of a and b.

Solution

$q = ab^p$

Taking logarithms: $\log q = \log ab^p$

$\log q = \log a + \log b^p$

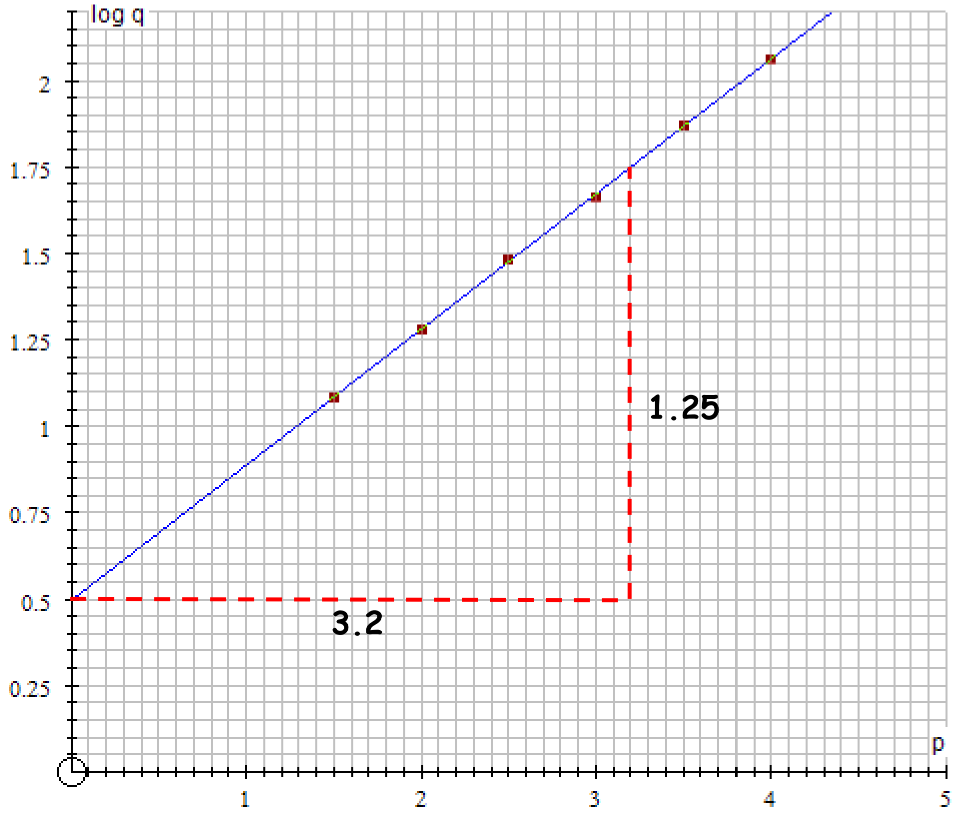
$\log q = \log a + p\log b$

If $\log q$ is plotted against p, this is the equation of a straight line with gradient $\log b$ and intercept $\log a$.

Again, this could be done using natural logs instead

Plot the values of $\log q$ against p:

p	1.5	2.0	2.5	3.0	3.5	4.0
q	12	19	30	46	74	116
$\log q$	1.08	1.28	1.48	1.66	1.87	2.06



Since the graph is approximately a straight line, the relationship $q = ab^p$ is an appropriate model.

Gradient = $\log b = \dfrac{1.25}{3.2} \Rightarrow b = 10^{1.25/3.2} \approx 2.5$

Intercept = $\log a = 0.5 \Rightarrow a = 10^{0.5} \approx 3.2$

The relationship is approximately $q = 3.2 \times 2.5^p$

You do not need to remember the details of what to plot and what to do with the gradient and intercept of the graph – all you need to do is to take logs of both sides of the suggested relationship and apply the laws of logarithms to obtain a relationship of the form $y = mx + c$, as shown above for each of the relationships $y = kx^n$ and $y = ka^x$. Once you have done this, you can see what you need to plot and how to find the values of the constants.

If the graph is not a straight line, then the suggested model is not an appropriate one (or perhaps the experimental results are not sufficiently accurate).

Section 3: Modelling curves

Exercise level 1

1. Two variables s and t are related by the formula $s = at^c$, where a and c are constants.
   (i) Show that this relationship can be written as $\log s = \log a + c \log t$.
   (ii) Explain why the model can be tested by plotting log s against log t.

   Values of s and t are recorded in an experiment.

s	9	13	16	18	20	22
t	5	10	15	20	25	30

   (iii) Plot the graph of log s against log t and use your graph to estimate the values of a and c.

2. Two variables a and b are related by the formula $b = mn^a$, where m and n are constants.
   (i) Show that this relationship can be written as $\ln b = \ln m + a \ln n$.
   (ii) Explain why the model can be tested by plotting ln b against a.

   In an experiment, the following values of a and b are obtained.

a	0.5	1.0	1.5	2.0	2.5	3.0	3.5
b	4.5	4.0	3.6	3.2	2.9	2.6	2.3

   (iii) Plot the graph of ln b against a and use your graph to estimate the values of m and n.

3. In an experiment, the temperature of a cooling jacket is measured in °C after t minutes, and the following data is found:

t minutes	0	3	6	10	14	20
θ°C	60	44.1	30.9	19.9	12.9	6.7

   The experimenter expects the data to fit a law of the form $\theta = ka^{-t}$.
   (i) Plot a graph of $\log \theta$ against t.
   (ii) Use your graph to find the law which the experimenter seeks.

Section 3: Modelling curves

Solutions to Exercise level 1

1. (i) $s = at^c$

Taking logarithms of both sides: $\log s = \log(at^c)$

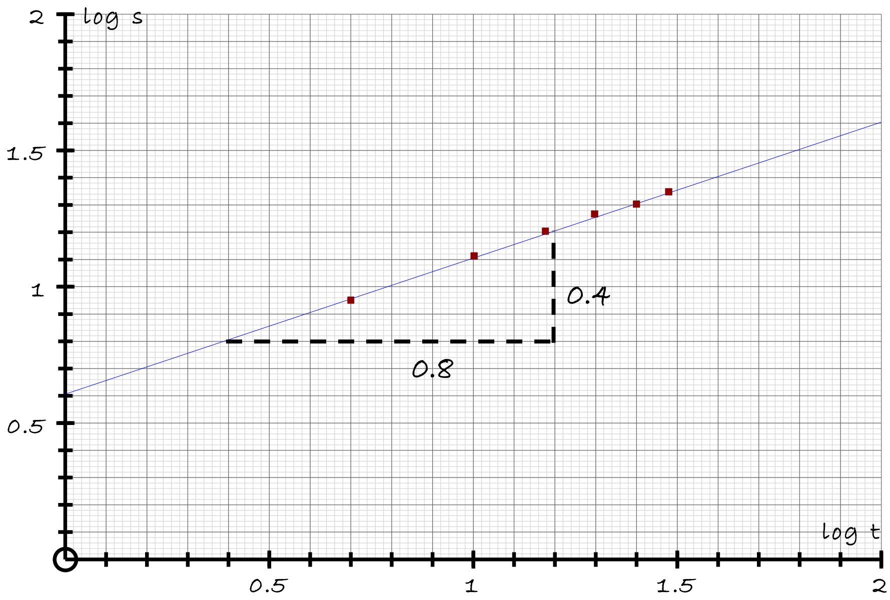
$$= \log a + \log t^c$$

$$= \log a + c \log t$$

(ii) Since log a and c are constants, the equation $\log s = \log a + c\log t$ is the equation of a straight line, in which the variables are log t and log s, and which has gradient c and intercept log a. So if the model is appropriate, plotting log s against log t will give an approximate straight line.

(iii)

s	9	13	16	18	20	22
t	5	10	15	20	25	30
log s	0.95	1.11	1.20	1.26	1.30	1.34
log t	0.70	1	1.18	1.30	1.40	1.48



Equation of graph is $\log s = \log a + c\log t$

Gradient $= \dfrac{0.4}{0.8} = 0.5$, so $c = 0.5$

Intercept $= 0.6$, so $\log a = 0.6 \quad \Rightarrow a = 10^{0.6} \approx 4$.

AQA AS Maths Exponentials and logs Exercise solutions

2. (i) $b = mn^a$

$\ln b = \ln(mn^a)$

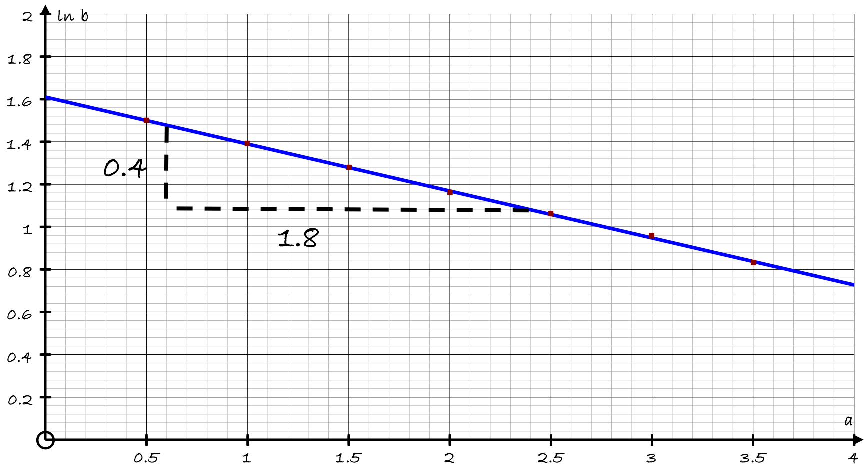
$= \ln m + \ln n^a$

$= \ln m + a\ln n$

(ii) The equation $\ln b = \ln m + a\ln n$ is the equation of a straight line, in which the variables are ln b and a, and which has gradient ln n and intercept ln m. So if the model is appropriate, then plotting ln b against a will give an approximate straight line graph.

(iii)

a	0.5	1.0	1.5	2.0	2.5	3.0	3.5
b	4.5	4.0	3.6	3.2	2.9	2.6	2.3
$\ln b$	1.50	1.39	1.28	1.16	1.06	0.96	0.83



Equation of graph is $\ln b = \ln m + a\ln n$

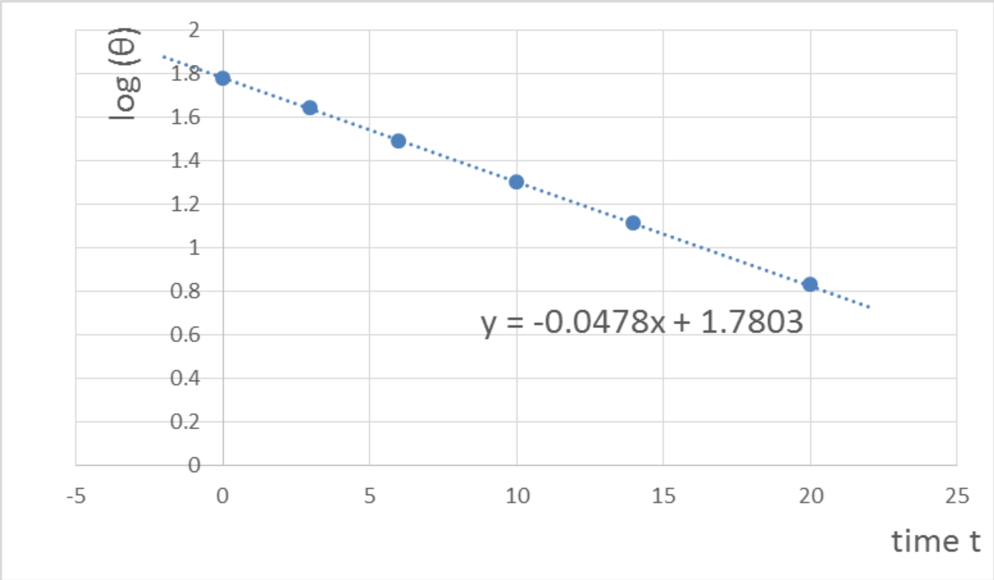
Gradient $= -\dfrac{0.4}{1.8} = -0.22$, so $\ln n = -0.22 \quad \Rightarrow n = e^{-0.22} \approx 0.8$

Intercept $= 1.6$, so $\ln m = 1.6 \quad \Rightarrow m = e^{1.6} \approx 5$

3. (i)

t minutes	0	3	6	10	14	20
θ °C	60	44.1	30.9	19.9	12.9	6.7
log(θ)	1.778151	1.644439	1.489958	1.298853	1.11059	0.826075

so the graph of log θ against t is:



(ii) The equation of the graph is $\log\theta = -t\log a + \log k$

Gradient $\approx -0.0478 = -\log a \Rightarrow a \approx 1.116$

Intercept $\approx 1.7803 = \log k \Rightarrow k \approx 60.3$

so the law is $\theta \approx 60.3 \times 1.116^{-t}$

Section 3: Modelling curves

Exercise level 2

1. The relationship between two variables x and y is believed to be of the form $y = kx^n$, where k and n are constants.

   In an experiment, the following values of x and y are recorded.

x	1	2	3	4	5	6	7
y	3	8	16	24	34	44	56

   (i) Plot the graph of ln y against ln x and explain why this tells you that the model $y = kx^n$ is appropriate.
   (ii) Use your graph to estimate the values of k and n.
   (iii) Estimate the value of y when $x = 10$.

2. The relationship between two variables p and q is believed to be of the form $q = ab^p$, where a and b are constants.

   In an experiment, the following values of p and q are recorded.

p	1.2	3.4	5.7	6.2	7.4	9.8
q	2.5	3.7	5.8	6.1	7.7	11.9

   (i) Plot the graph of log q against p, and explain why this tells you that the model $q = ab^p$ is appropriate.
   (ii) Use your graph to estimate the values of a and b.
   (iii) Estimate the value of q when $p = 12$.

3. An engineer finds the following data in an investigation into the breaking strain of a bridge cable when it is subjected to different levels of a particular treatment. The data is given in suitable, but undefined, units.

Treatment x	0.2	0.4	0.6	0.8	1
Breaking strain y	0.07	0.33	0.77	1.43	2.25

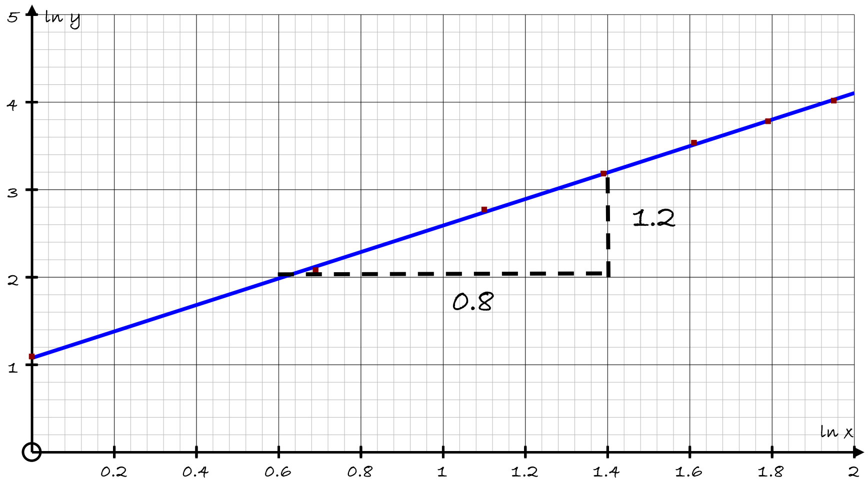
   (i) Sketch a graph of y against x, and suggest a form for a rule connecting them.
   (ii) Write your rule in terms of logarithms, and hence plot a suitable graph and use it to find approximately the law.

Section 3: Modelling curves

Solutions to Exercise level 2

1. (i)

x	1	2	3	4	5	6	7
y	3	8	16	24	34	44	56
$\ln x$	0	0.69	1.10	1.39	1.61	1.79	1.95
$\ln y$	1.09	2.08	2.77	3.18	3.53	3.78	4.02



$y = kx^n$

$\ln y = \ln k + n \ln x$

This is the equation of a straight line, with variables ln y and ln x.
Since the points form an approximate straight line, the model is appropriate.

(ii) $\ln y = \ln k + n \ln x$ is the equation of a straight line with gradient n and intercept ln k.

Gradient $= \frac{1.2}{0.8} = 1.5 \quad \Rightarrow n = 1.5$

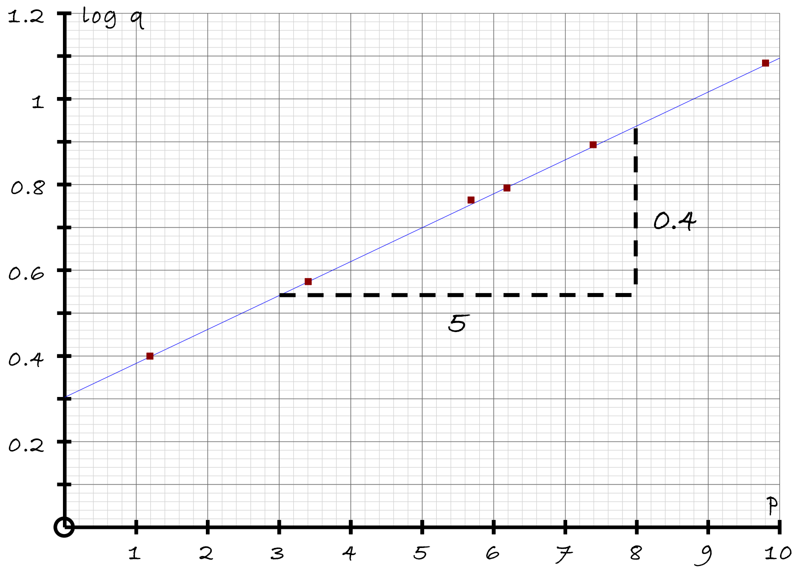
Intercept $= 1.1$ so $\ln k = 1.1 \quad \Rightarrow k = e^{1.1} \approx 3$

(iii) $y = 3x^{1.5}$

When $x = 10$, $y = 3 \times 10^{1.5} \approx 95$

2. (i)

p	1.2	3.4	5.7	6.2	7.4	9.8
q	2.5	3.7	5.8	6.1	7.7	11.9
$\log q$	0.40	0.57	0.76	0.79	0.89	1.08



$q = ab^p$

$\log q = \log a + p \log b$

This is the equation of a straight line, with variables log q and p.

Since the points form an approximate straight line, the model is appropriate.

(ii) $\log q = \log a + p \log b$ is the equation of a straight line with gradient log b and intercept log a.

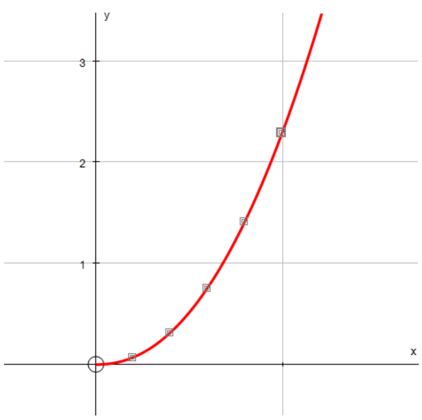
Gradient $= \dfrac{0.4}{5} = 0.08$, so $\log b = 0.08 \quad \Rightarrow b = 10^{0.08} \approx 1.2$

Intercept $= 0.3$, so $\log a = 0.3 \quad \Rightarrow a = 10^{0.3} \approx 2$

(iii) $q = 2 \times 1.2^p$

When p = 12, $q = 2 \times 1.2^{12} \approx 17.8$

3. (i)



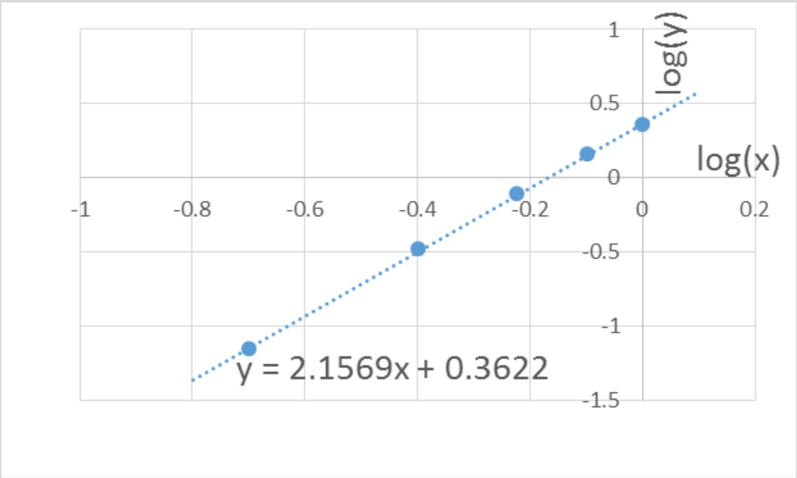
The graph appears to be a power curve, so suggest a law of form $y = kx^a$.

(ii) $y = kx^a$

$\Rightarrow \log y = a\log x + \log k$

so plot a graph of log y against log x.

Log x	-0.69897	-0.39794	-0.22185	-0.09691	0
Log y	-1.1549	-0.48149	-0.11351	0.155336	0.352183



Gradient $\approx 2.1569 = a$

Intercept $\approx 0.3622 = \log k \Rightarrow k \approx 2.303$

so the law is approximately $y = 2.30x^{2.16}$

AQA AS Maths Exponentials and logarithms

Topic assessment

1. Write as a single logarithm:
   (i) $2\log a + 3\log b$ (ii) $\log x - 3\log y + 4\log z$ [4]

2. Express the following in terms of log p, log q and log r.
   (i) $\log \frac{pq}{r}$ (ii) $\log \frac{\sqrt{p}}{r^2}$ [4]

3. Solve the following equations;
   (i) $2^x = 7$ (ii) $3^{2x} = 5$ [4]

4. Solve the equations
   (i) $2e^x = 3e^{-x} + 5$ [3]
   (ii) $\ln(2x+1) = \ln x + 2$ [3]
   giving your answers in exact form.

5. Alice puts £500 in a savings account, at a fixed interest rate of 5% per year, when her grandson Harry is born. Interest is added to the account on Harry's birthday each year. The amount, P, in the account after n years is given by:
   $$P = 500 \times 1.05^n$$
   How old will Harry be when the amount in the savings account first exceeds £1000? [4]

6. The number N of rabbits in a colony after t years is modelled by $N = 20 \times 2^{0.8t}$.
   (i) How many rabbits are in the colony after 5 years? [2]
   (ii) A biologist suggests that due to limited resources, this model will no longer be appropriate when N reaches 2000. For how many years will this model be appropriate? [3]

7. The temperature T°C of the water in a kettle t minutes after boiling is modelled by the equation $T = 20 + 80e^{-0.5t}$.
   (i) What is the initial temperature of the water? [1]
   (ii) Find the temperature of the water after 5 minutes. [2]
   (iii) Find the time at which the temperature of the water is 30°C. [3]
   (iv) Find the initial rate of cooling, and the rate of cooling after 2 minutes. [3]
   (v) What will be the long-term temperature of the water? [1]

8. In an experiment, the number of bacteria, N, in a culture was estimated at time t days after the measurements started.
   The results were as follows:

t	1	2	3	4	5	6
N	120	170	250	400	620	910

It is believed that the relationship between N and t can be expressed in the form

$$N = ab^t$$

where a and b are constants.

(i) Explain how this can be tested by plotting log N against t. [2]

(ii) Make out a table of values of log N and draw the graph. [3]

(iii) Use your graph to estimate the values of a and b. [3]

(iv) Estimate the number of bacteria present after 20 days. State, with a reason, whether your estimate is likely to be a good one. [2]

9. It is believed that two quantities, x and y, are connected by a relationship of the form $y = kx^n$, where k and n are constants.

In an experiment, the following data were produced.

x	5	10	15	20	25	30	35
y	9	24	48	69	102	131	166

(i) Explain how the form of the relationship can be tested by plotting log y against log x. [2]

(ii) Make out a table of values of log x and log y and plot the graph. [3]

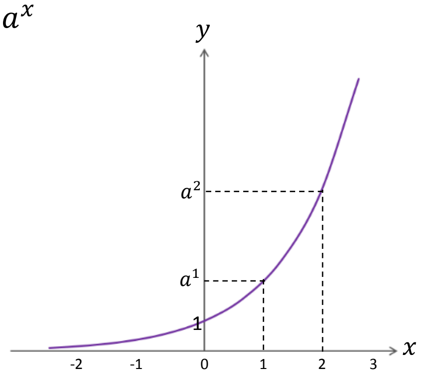
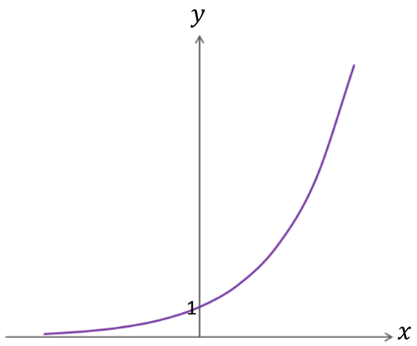
(iii) Use your graph to estimate the values of k and n. [3]

Total 55 marks

Summary sheet: Exponentials and logarithms

F1 Know and use the function a^x and its graph, where a is positive
Know and use the function e^x and its graph
F2 Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications
F3 Know and use the definition of $\log_a x$ as the inverse of a^x , where a is positive and $x \geq 0$
Know and use the function ln x and its graph
Know and use ln x as the inverse function of e^x
F4 Understand and use the laws of logarithms: $\log_a x + \log_a y = \log_a(xy)$, $\log_a x - \log_a y = \log_a(x/y)$,
$k\log_a x = \log_a x^k$ (including for example k = -1 and k = ½)
F5 Solve equations of the form $a^x = b$
F6 Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y
F7 Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models

Exponential Functions

Graph:	Points to notice	Tips
$y = a^x$ 	• Always crosses the y-axis at 1 ($a^0 = 1$) • The x-axis is an asymptote as you can never get a y-value of 0 ($a^x \neq 0$)	If you don't remember what the graph looks like, try substituting a with a number (e.g. use 3^x) and find some points. Plot them to get an idea of what the graph looks like.
$y = e^x$ 	• The graph looks the same as you have just replaced a with e.	Find a few points to see what the graph looks like.

The gradient of e^{kx}

If: $y = e^{kx}$

then: $\frac{dy}{dx} = ke^{kx}$

Remember that $\frac{dy}{dx}$ is the gradient function and so this is the gradient of e^{kx}

Summary sheet: Exponentials and logarithms

Logs

What does a log mean? E.g. $\log_2 16$

This is a log with base 2.

$\log_2 16$ means "***What power do I raise 2 to, to get 16?***" (i.e. $2^? = 16$)
The answer is 4 (because $2^4 = 16$)

$\therefore \log_2 16 = 4$

You can summarise like this:

$$\boldsymbol{y = \log_a x \iff x = a^y}$$
(for $a > 0$ and $x > 0$)

This means that "log to the base n" and "n to the power of" are the opposite (inverse) of each other and will undo each other (cancel each other out).

e.g. $2^{\log_2 5} = 5$ **and** $\log_2(2^5) = 5$

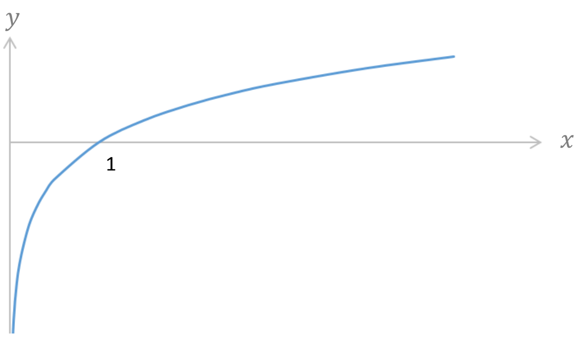
Cancel each other out   Cancel each other out

ln (the natural log)

ln has a base of e, and so ln and e are the opposite (inverse) of each other and will undo each other (cancel each other out).

e.g. $e^{\ln 7} = 7$ **and** $\ln(e^7) = 7$

The graph of $\ln x$

<table>
<tr><th>Graph:</th><th>Points to notice</th></tr>
<tr><td>$y = \ln x$

</td><td>

- Always crosses the x-axis at 1 ($\ln 1 = 0$)
- The y-axis is an asymptote as you cannot get an answers for $\ln 0$ (try it on your calculator, you will get an error - you can't raise e to any power and get the answer 0)

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</table>

Summary sheet: Exponentials and logarithms

The laws of logs

You need to learn, and know how to use, the following laws of logs:

Law	Example
$\mathbf{\log(x) + \log(y) = \log(xy)}$	$\log(2) + \log(5) = \log(2 \times 5) = \log(10)$
$\mathbf{\log(x) - \log(y) = \log\left(\frac{x}{y}\right)}$	$\log(12) - \log(3) = \log\left(\frac{12}{3}\right) = \log(4)$
$\mathbf{\log(x^k) = k\log(x)}$	$\log(5^2) = 2\log(5)$
$\mathbf{\log(1) = 0}$	$\log_{27}1 = 0$

All of the laws are true for any base (including base e, i.e. ln).

Solve equations of the form $a^x = b$

To solve this type of equation you need to bring the x down from the power, so you will use the 3rd law:

$$\log(x^k) = k\log(x)$$

Step 1: Take the log of both sides.
Step 2: use the 3rd rule to bring the power to the front.
Step 3: Solve the equation as normal.

e.g. Solve the equation: $3^{x-5} = 2$

Step 1: Take the log of both sides: $\log(3^{x-5}) = \log(2)$

Step 2: use the 3rd rule: $(x-5)\log3 = \log2$

Step 3: Tidy up and solve:

$$(x-5) = \frac{\log2}{\log3}$$

$$(x-5) = 0.6309$$

$$x = 0.6309 + 5$$

$$\mathbf{x = 5.6309}$$

Summary sheet: Exponentials and logarithms

Logarithmic Graphs

When you have a relationship of the form $y = kx^n$ or $y = ab^x$ it can be tricky to find the parameters $(k, a\ and\ b)$ from the curve. Taking logs of both sides turns the relationship into a straight line and makes finding the parameters easier.

	$y = kx^n$	$y = ab^x$
Original:	$y = kx^n$	$y = ab^x$
Take logs of both sides:	$\log(y) = \log(kx^n)$	$\log(y) = \log(ab^x)$
Tidy up using laws of logs:	$\log(y) = \log(k) + \log(x^n)$	$\log(y) = \log(a) + \log(b^x)$
	$\log(y) = \log(k) + n\log(x)$	$\log(y) = \log(a) + x\log(b)$
	$\log(y) = n\log(x) + \log(k)$	$\log(y) = x\log(b) + \log(a)$
You now have a straight line (of the form $y = mx + c$) where:	gradient $= n$ intercept $= \log k$	gradient $= \log b$ intercept $= \log a$

For either of the above you can plot the graph and find the gradient and the intercept.

e.g. you have been given data for x and y and it is thought that the relationship is of the form $y = kx^n$. Verify this and find the approximate values of k and n.

Data:

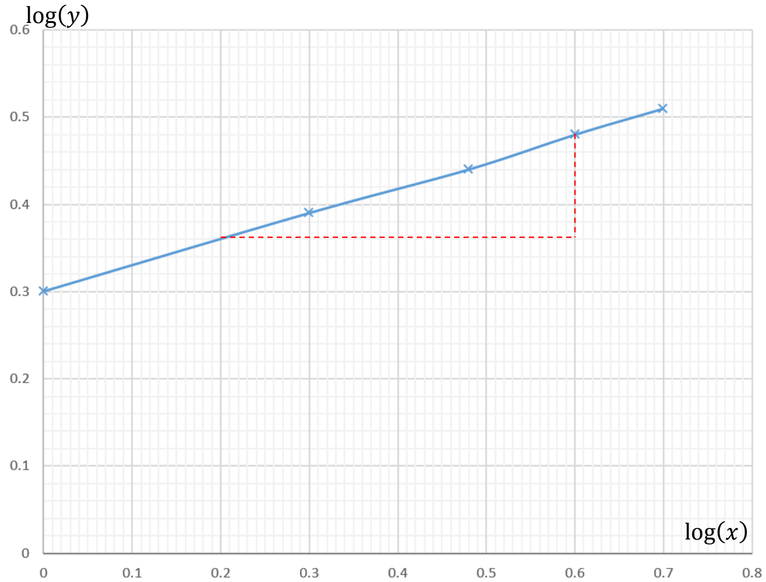
x	1	2	3	4	5
y	2	2.46	2.78	3.03	3.24

Take logs of each side, as shown above, to get: $\log(y) = n\log(x) + \log(k)$

You need to plot $\log(y)$ against $\log(x)$ so first of all find the values of $\log(y)$ and $\log(x)$:

x	1	2	3	4	5
$\log(x)$	0	0.3	0.48	0.6	0.7
y	2	2.46	2.78	3.03	3.24
$\log(y)$	0.3	0.39	0.44	0.48	0.51

Now plot the graph of $\log(y)$ against $\log(x)$:



Gradient $(n) = \frac{0.48-0.36}{0.6-0.2} = 0.3$

Intercept $(\log(k)) = 0.3$

$\therefore k = 10^{0.3} = 1.99\ (\approx 2)$

You have found that the relationship is approximately:

$$y = 2x^{0.3}$$

Summary sheet: Exponentials and logarithms

e.g. you have been given data for x and y and it is thought that the relationship is of the form $y = ab^x$. Verify this and find the approximate values of a and b.

Data:

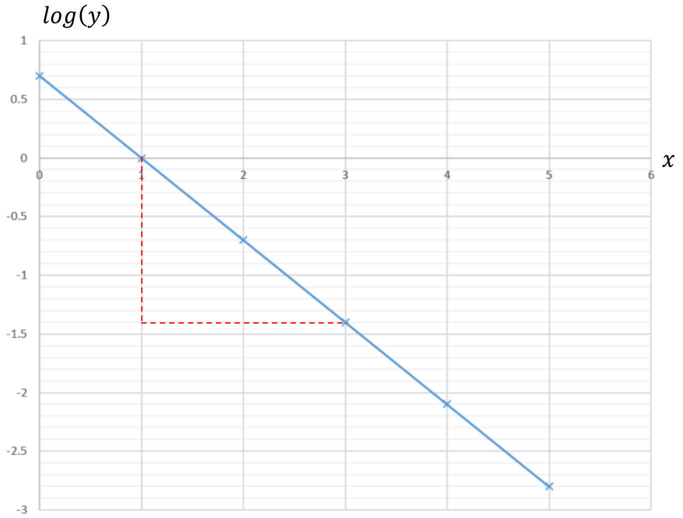
x	0	1	2	3	4	5
y	5	1	0.2	0.04	0.008	0.0016

Take logs of each side, as shown above, to get: $\log(y) = x\log(b) + \log(a)$

You need to plot $\log(y)$ against x so first of all find the values of $\log(y)$:

x	0	1	2	3	4	5
y	5	1	0.2	0.04	0.008	0.0016
$\log(y)$	0.7	0	-0.7	-1.4	-2.1	-2.8

Now plot the graph of $\log(y)$ against x:



Gradient $(\log(b)) = \frac{-1.4-0}{3-1} = -0.7$

$\therefore b = 10^{-0.7} \approx 0.2$

Intercept $(\log(a)) = 0.7$

$\therefore a = 10^{0.7} = 5.01\ (\approx 5)$

You have found that the relationship is approximately:

$$y = 5 \times 0.2^x$$